Large Aggregative Games with Heterogeneous Players

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Abstract

We study large games played by heterogeneous agents whose payoffs depend on the aggregate action and provide novel equilibrium selection and comparative statics results. Regarding equilibrium selection, we establish the equivalence between potential maximization and the global games selection in supermodular games, and characterize the uniquely selected equilibrium as the strategy profile that maximizes the ex-ante expected payoffs of a player with downward-biased beliefs about the aggregate action. To obtain our equivalence result we show that (i) payoffs in an aggregative potential game must be *quasilinear* and (ii) beliefs in the global game must satisfy a *generalized Laplacian property* linking (weighted) average beliefs to the uniform distribution. We present comparative statics results that rely on average rather than pointwise conditions on payoffs and use them to illustrate how heterogeneity affects equilibrium levels of the aggregate action.

Keywords: aggregative games, potential games, global games, comparative statics, noise-independent selection, Laplacian beliefs

1 Introduction

Large games in which individual payoffs depend on the aggregate behavior in the population are ubiquitous in economics and finance.[1] An incomplete list includes

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[1]These games are referred in the literature as (linear) aggregative games (Jensen, 2018), average-action games (Morris and Shin, 2003), games with aggregation (Dubey et al., 2006) or aggregate games (Martimort and Stole, 2012).

models of market competition (aggregate output), macroeconomic coordination (average search effort), public goods and externalities (sum of contributions), technology adoption (average investment), as well as binary-action games where payoffs depend on the fraction of agents adopting each action such as platform choice, bank runs, currency crises or games of regime change. In all these economic phenomena, agent heterogeneity is a defining characteristic of the environment (e.g., agents may differ in costs, preferences, productivity or endowments), and understanding its effects should be an integral part of the analysis.

A typical feature of these models is the presence of multiple equilibria, which has been tackled by using equilibrium selection rules that focus on a particular equilibrium or by resorting to comparative statics that apply to the set of equilibria. However, the introduction of heterogeneity complicates equilibrium analysis in two main ways. First, heterogeneity limits the appeal of popular selection rules, such as those based on introducing incomplete information (e.g., global games), due to the inability to characterize the selected equilibrium or because of the lack of clear economic content behind the selection. Second, most of the existing comparative statics results rely on monotonicity restrictions at the individual level, which may not apply to heterogeneous models where agents have divergent interests.

We address these limitations by providing new equilibrium characterization and comparative statics results for large aggregative games with heterogeneous payoff types. Specifically, we establish the equivalence between two commonly used equilibrium selection rules in games with strategic complementarities: the global games selection (Carlsson and van Damme, 1993; Frankel et al., 2003) and potential maximization (Monderer and Shapley, 1996). We also characterize the selected equilibrium and give economic content to the selection by showing that maximizing potential coincides with maximizing the ex-ante payoffs of an agent with marginal beliefs, who thinks that she is the lowest type contributing to the aggregate action. In addition, we generalize existing results on monotone comparative statics by replacing conditions that all player types must satisfy (e.g., strategic complementarities) with weaker restrictions that only apply on average. In particular, we identify conditions on average payoffs under which the set of equilibrium aggregate actions 'moves up' after a change in parameters or in the distribution of types.

The games we study have payoffs that depend on the player's action $a \in \mathbb{R}$, the aggregate action $\bar{a}$ and the player's type w. We obtain the equilibrium characterization by uncovering two key properties. First, we determine the payoff structure required for the game to be a potential game. Second, we pin down average player beliefs about the aggregate action in global games with heterogeneous payoffs.

Section 3 shows that potential games must exhibit *quasilinear payoffs*, which take on the following form: $U(a, \bar{a}, w) = au(\bar{a}) + v(a, w)$. This payoff structure figures prominently in the economics literature, such as in the classic models of Cournot competition, Diamond's search model, Tullock contests and most binary-action games, just to name a few.[2] Potential games are defined by the existence of a single function of the strategy profile (i.e., the potential function) that captures the change in individual payoffs of any player following a change in her strategy. Because the potential function must reflect individual payoff changes for all types, its existence imposes strong symmetry restrictions on the payoff impact of the aggregate action, which are only satisfied by quasilinear payoffs.

Section 4 presents a characterization of beliefs in global games, which we call the *Generalized Laplacian Property* since it generalizes to many-action, heterogeneous-player games both the Laplacian property of homogeneous binary-action games (Morris and Shin, 2003) and its counterpart for binary-action heterogeneous games (Sakovics and Steiner, 2012). In a global game, agents receive noisy signals about some payoff parameter and, because of this, face uncertainty about the aggregate action. The property states that the weighted average belief about the aggregate action is given by the uniform distribution, where the weights are proportional to the contribution of each type to the aggregate action.

We use the generalized Laplacian property to show that if payoffs are quasilinear then, as noise in the global game vanishes, the change in a player's expected payoffs after switching actions converges to the (infinitesimal) change in potential of the complete information game. This implies that equilibrium in the global game converges to the strategy profile that maximizes potential. We characterize the selected equilibrium by pinning down the functional form of the potential function,

[2]Models with quasilinear payoffs include externalities (Dybvig and Spatt, 1983), technology adoption (Farrell and Saloner, 1985; Katz and Shapiro, 1986), contests (Tullock, 1980; Cornes and Hartley, 2005), common resources (Dasgupta and Heal, 1979), macroeconomic search (Diamond, 1982), cost-sharing (Moulin, 1990), bank runs (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005), currency crises (Obstfeld, 1986; Morris and Shin, 1998; Guimaraes and Morris, 2007), tax evasion (Bassetto and Phelan, 2008), regime change and coordination games (Sakovics and Steiner, 2012), crime waves (Bond and Hagerty, 2010), and blockchain (Abadi and Brunnermeier, 2018).

and provide economic content behind the selection by deriving a dual representation of potential as the ex-ante payoffs of a player with marginal beliefs. This interpretation allows for a comparison with other selection rules such as Pareto dominant equilibrium, which maximizes the ex-ante expected payoff given correct beliefs, and answers the open question about the meaning of maximizing potential in the context of aggregative games.[3]

In addition to our characterization result, we also present robust comparative statics for games with quasilinear payoffs that apply to the set of equilibrium aggregate actions (Section 5). We show that the set moves up following a change in parameters or in the distribution of payoff types if *average payoffs* across types satisfy a monotonicity restriction (increasing differences). In contrast, the existing literature relies on monotonicity restrictions on payoffs that must apply pointwise for all types (Milgrom and Shannon, 1994) or, alternatively, on monotonicity restrictions on best responses instead of on primitives of the game (Acemoglu and Jensen, 2010; Camacho et al., 2018). We obtain our results by recasting the problem of finding equilibrium as a fixed point problem over the set of aggregate actions instead of full strategy profiles. We highlight the differences with existing results by showing that the smallest and largest equilibrium aggregate actions go up with parameters even if individual incentives are not monotone, that is, even if the game is not supermodular, which allows some agents to exhibit decreasing best responses. Accordingly, our results can be suitable for games where agents have diverging interests. Moreover, our comparative statics results regarding the distribution of types only rely on average restrictions on the heterogeneous component of payoffs $v(a, w)$, leading to a tractable analysis of the impact of heterogeneity on aggregate behavior.

Overall, the paper shows that in games with quasilinear payoffs one can obtain results by replacing pointwise conditions by average conditions on payoffs and beliefs. The paper contributions, which also include a novel definition of potential for games with continuous actions and types, touch upon many areas of economic theory. Accordingly, after presenting the main definitions and results, we discuss the related literature in Section 6.

[3] In their paper introducing potential games, Monderer and Shapley (1996) openly ask about the meaning of potential maximization (bottom of page 125, square brackets added for clarification):

"This raises the natural question about the economic content (or interpretation) of P^* [potential maximizer]: What do the firms [players] jointly try to maximize? We do not have an answer to this question."

2 Large Aggregative Games

There is a continuum of players of measure one. Each agent is endowed with a type $w \in [\underline{w}, \overline{w}]$ and the mass of agent types is distributed in the population according to cdf F with density f. They simultaneously choose an action from the set $A \subset \mathbb{R}_+$, which can be any compact, countable union of single points and closed intervals. Leading examples include finite action and continuous action games. To simplify notation, we normalize actions so that the lowest action in A is set to zero.[4] Let $a_{max} := \max A$ denote the highest action in A.

The payoffs of a player of type w choosing action a are given by $U(a, \bar{a}, \theta, w)$, where $\bar{a} \in [0, a_{max}]$ denotes the average of players' actions in the population and θ is a common parameter that belongs to the closed interval $\Theta \subset \mathbb{R}$. We assume that $U(\cdot)$ is Lipschitz continuous, differentiable with respect to $\bar{a}$ and bounded.[5] In addition, let

$$\Delta U(a, a', \bar{a}, \theta, w) = U(a, \bar{a}, \theta, w) - U(a', \bar{a}, \theta, w)$$

denote the payoff differences between choosing a and a' for a player of type w.

A strategy profile is given by a measurable mapping $\alpha : [\underline{w}, \overline{w}] \to A$. Let $\mathcal{A}$ denote the set of measurable functions α.[6] The average or aggregate action of players with types in any measurable subset $W \subseteq [\underline{w}, \overline{w}]$ under profile α is given by

$$\bar{\alpha}(\alpha, W) = \int_{\underline{w}}^{\overline{w}} \alpha(w) dF(w | w \in W), \tag{1}$$

Abusing notation we use $\bar{\alpha}(\alpha)$ to denote the aggregate action in the whole population $\bar{\alpha}(\alpha, [\underline{w}, \overline{w}])$. Accordingly, for any strategy profile $\alpha \in \mathcal{A}$, the payoffs of a player of type w are given by $U(\alpha(w), \bar{\alpha}(\alpha), \theta, w)$.

Formally, a game is given by the tuple $\Gamma_\theta = \{F, A, \theta, U\}$. Both F and θ are common knowledge, i.e., Γ_θ is a game of complete information. A Nash equilibrium

[4] If $\min A \neq 0$ we can always redefine the set of actions to be $A' = \{a - \min A, a \in A\}$.

[5] By continuity in own action a we mean at interior points of A. Our results can be extended to payoffs that exhibit discontinuities at certain values of $\bar{a}$ or θ, as is the case in regime change models (see Serrano-Padial (2018) for details).

[6] The restriction to measurable strategies ensures that payoffs are well defined. It is without loss of generality in supermodular games, which exhibit equilibrium strategies that are monotone functions from a measurable subset of $\mathbb{R}$ to A and thus are measurable. However, it may be restrictive in general since it imposes that all agents of the same type use the same (pure) strategy.

(NE) of the game is a strategy profile $\alpha^* \in \mathcal{A}$ satisfying

$$\alpha^*(w) \in \arg\max_{a \in A} U(a, \bar{\alpha}(\alpha^*), \theta, w) \text{ for all } w \in [\underline{w}, \overline{w}].$$

We use Γ to denote the family of games $\{\Gamma_\theta, \theta \in \Theta\}$.

3 Quasilinear Payoffs and Potential

In this section we show the equivalence between quasilinear payoffs and potential games. First, we define quasilinearity and potential.

Definition 1. Payoffs are *quasilinear* if there exist functions u, v such that

$$U(a, \bar{a}, \theta, w) = au(\bar{a}, \theta) + v(a, \theta, w),$$

or, more generally, if U can be expressed as $c(\theta, w)\big(au(\bar{a}, \theta) + v(a, \theta, w)\big) + u_0(\bar{a}, \theta, w)$ with $c(\theta, w) > \xi > 0$ for all θ, w.

Most models with quasilinear payoffs deal with binary-action games ($A = \{0, 1\}$) or with continuous-action games ($A = [0, a_{max}]$). Binary-action games include most coordination games such as regime change models in which the probability of the regime failing is a function of the fraction attacking the regime. Perhaps the leading example of continuous-action games is the model of Cournot competition with heterogeneous cost functions, where a is individual output, u represents inverse demand and v are production costs.[7]

Quasiliniearity in binary-action games translates into payoff differences being *separable* in aggregate action and type, as defined in Serrano-Padial (2018). That is, $\Delta U(1, 0, \bar{a}, \theta, w) = u(\bar{a}, \theta) + v_\Delta(\theta, w)$, where $v_\Delta(\theta, w) = v(1, \theta, w) - v(0, \theta, w)$.

We will use the following two examples featuring continuous actions to provide intuition about our results. The first one is about strategic complementarities in investment, and it will be used to illustrate how to derive the equilibrium selection rules and characterize the selected equilibrium. The second example is about negative externalities, e.g., due to congestion, and will highlight the usefulness of our comparative statics results.

[7]See footnote 2 for additional examples.

Example 1 (Investment Game). *Consider an economy populated by a continuum of heterogeneous firms choosing how much to invest in a new technology. Each firm chooses investment level* $a \in [0, a_{max}]$. *Unit returns on investment, given by* $u(\bar{a}, \theta)$, *are increasing in the degree of technology adoption* $\bar{a}$ *and in the quality of the technology* θ. *Investment costs are quadratic and inversely proportional to the firm's productivity type* $w \in [\underline{w}, \overline{w}] \subset \mathbb{R}_{++}$, *which is distributed according to* F *in the population with* $Ew = 1$. *Specifically, payoffs are given by*

$$U(a, \bar{a}, \theta, w) = au(\bar{a}, \theta) - \frac{a^2}{2}\frac{1}{w}. \tag{2}$$

Example 2 (Negative externalities). *A continuum of agents choose their individual consumption level* $a \in [0, a_{max}]$ *of an exhaustible/congestible good (e.g., roads, cell-phone bandwidth, natural resources). Payoffs are given by the benefit from usage, which depends on type* w *and a common attribute* θ *of the good, minus (linear) costs, which increase with average consumption* $\bar{a}$. *Specifically,*

$$U(a, \bar{a}, \theta, w) = b(a, \theta, w) - c(\bar{a}, \theta)a,$$

where b, c *are differentiable,* b *is strictly concave in* a *and* c *is increasing in* $\bar{a}$.

We next provide a formal definition of potential for the class of large aggregative games defined above. The original definition by Monderer and Shapley (1996) applies to finite games, and involves the existence of a mapping from strategy profiles to the real line, called the potential function, whose change after a single player switches strategies coincides with the player's payoff change. Since the game $\Gamma_\theta = \{F, A, \theta, U\}$ exhibits a continuum of players and types, we account for the fact that any single type switching strategies has no measurable impact on the strategy profile by defining potential as a function whose infinitesimal change after a single type switches strategies coincides with the type's change in payoffs.[8] We do so by using a mixture distribution that places a positive mass on the type switching actions and take the

[8] Existing definitions of potential focus on finite games (e.g., Monderer and Shapley, 1996) or population games with a discrete distribution of types (e.g., Sandholm, 2009). We could accommodate finite types $\{1, \cdots, N\}$ by setting $[\underline{w}, \overline{w}] = [0, 1]$ and $F = U[0, 1]$. Then, we would partition $[0, 1]$ into N subintervals such that each interval i has length equal to the mass of discrete type i and assign the payoff function of discrete type i to all w in interval i. The proofs would need to be adjusted to allow for v to be discontinuous in w at the boundaries of each of the intervals.

limit of the change in potential as this mass goes to zero, while keeping the actions of all other types fixed. Let $\delta(w)$ denote the Dirac delta distribution that places all the probability mass on type w and $\mathcal{F}$ the space of distribution functions on $[\underline{w}, \overline{w}]$.

Definition 2 (Potential)**.** The game Γ_θ is a *potential game* if there exists a functional $V : \mathcal{A} \times \mathcal{F} \times \Theta \to \mathbb{R}$ satisfying the following two conditions:

1. Let F_w^ε denote the mixture distribution $(1-\varepsilon)F + \varepsilon\delta(w)$ for $\varepsilon \in (0,1)$. For all w, all $a \in A$, and all $\alpha \in \mathcal{A}$

$$\lim_{\varepsilon \to 0} \frac{V(\alpha', F_w^\varepsilon, \theta) - V(\alpha, F_w^\varepsilon, \theta)}{\varepsilon} = \Delta U(a, \alpha(w), \bar{\alpha}(\alpha), \theta, w), \tag{3}$$

   where $\alpha'(w) = a$ and $\alpha'(w') = \alpha(w')$ for all $w' \neq w$.

2. Let $F_{ww'}^{\varepsilon\epsilon}$ denote the mixture distribution $(1-\varepsilon-\epsilon)F + \varepsilon\delta(w) + \epsilon\delta(w')$ for $\varepsilon, \epsilon \in (0,1)$. For all w, w', and all $\{\alpha_i\}_{i=1}^4$ such that $\alpha_i(w'') = \alpha_{i+2}(w'')$ if $w'' \neq w$ for $i = 1,2$ and $\alpha_i(w'') = \alpha_{i+1}(w'')$ if $w'' \neq w'$ for $i = 1,3$,

$$\lim_{\epsilon \to 0} \lim_{\varepsilon \to 0} \frac{(V(\alpha_4, F_{ww'}^{\varepsilon\epsilon}, \theta) - V(\alpha_3, F_{ww'}^{\varepsilon\epsilon}, \theta)) - (V(\alpha_2, F_{ww'}^{\varepsilon\epsilon}, \theta) - V(\alpha_1, F_{ww'}^{\varepsilon\epsilon}, \theta))}{\varepsilon\epsilon} \tag{4}$$

   exists and coincides with the limit obtained by exchanging the order of limits.

We call $V(\cdot, F, \theta)$ the potential function of Γ_θ. Similarly, Γ_θ is a *weighted* potential game if there exists a function $\psi(w,\theta) > \zeta > 0$ such that the game $\{F, A, \theta, \tilde{U}\}$ with payoffs given by $\tilde{U}(a, \alpha, \theta, w) = \psi(w,\theta)U(a, \alpha, \theta, w)$ is a potential game. The potential of $\{F, A, \theta, \tilde{U}\}$ is the weighted potential of Γ_θ.

The first condition in Definition 2 is the mentioned adaptation of potential to the case of continuous actions and types while the second condition is akin to assuming differentiability of potential with respect to two types switching actions, adapted to the use of mixture distributions. The use of mixture distributions is especially convenient in games where payoffs only depend on the aggregate action, since they uniformly converge to the original distribution of types and have an intuitive way to capture the marginal effect of a change in the strategy of a finite number of types.

Our first result shows that quasilinearity is a necessary and sufficient condition for the existence of weighted potential, and provides the functional form of the potential function. Abusing notation, since the distribution of types F is a primitive of the family of games Γ, we denote the potential function $V(\alpha, F, \theta)$ by simply $V(\alpha, \theta)$. All proofs are relegated to Appendix B.

Proposition 1. *The game Γ_θ is a weighted potential game if and only if payoffs are quasilinear. In addition, the following functional represents the weighted potential of Γ_θ:*

$$V(\alpha, \theta) = \int_0^{\bar{a}(\alpha)} u(z, \theta)dz + \int_{\underline{w}}^{\overline{w}} v(\alpha(w), \theta, w)dF(w). \tag{5}$$

The necessity of quasilinear payoffs is brought about by symmetry restrictions imposed by the existence of potential. Roughly speaking, for a single function to reflect payoff differences for all types, two things must happen. First, the payoff function must exhibit symmetry with respect to the aggregate action across types. This leads to separability of payoffs into two components, one associated with the aggregate action and the other with the player type. Second, when a type switches actions the infinitesimal change in the aggregate action only depends on the difference between the two actions and not on the value of the player's action before the switch. Accordingly, the change in potential and thus the change in the player payoffs associated with such a change in the aggregate action must not depend on the initial action either. This implies that the payoff component associated with the aggregate action must be linear in own action. In other words, in aggregative games, quasilinear payoffs are a consequence of *externality symmetry* of action changes, a property of payoffs in potential games identified by Sandholm (2009) for the case of finite actions and types. For the continuous-action case, externality symmetry translates into equal cross-partial derivatives:

$$\frac{\partial^2 U(a, \bar{a}, \theta, w)}{\partial a \partial \bar{a}} = \frac{\partial^2 U(a', \bar{a}, \theta, w')}{\partial a \partial \bar{a}} \text{ for all } a, a', \bar{a}, w \text{ and } w'.$$

In what follows we assume that the assumption of Lipschitz continuity and boundedness not only applies to U but also to u and v.

4 Equilibrium Selection

The characterization of payoffs and the potential function in aggregative potential games allows us to establish the equivalence between two commonly used equilibrium selection rules: potential maximization and the global games selection. First, we introduce potential maximization and show that it is associated with finding the NE that maximizes the ex-ante payoffs of an agent with marginal beliefs, i.e., who thinks that types lower than hers do not contribute to the aggregate action. Second, we show that the potential maximizing NE is essentially unique if the game is supermodular. Finally, after introducing the global games selection, we show that both selection rules coincide in supermodular games with quasilinear payoffs.

4.1 Potential Maximization

This section introduces the notion of marginal beliefs and conveys two results. Proposition 2 shows that a strategy profile maximizing potential exists and maximizes the ex-ante payoffs of an agent with marginal beliefs. Proposition 3 establishes that there is an essentially unique potential maximizer in supermodular games (Milgrom and Roberts, 1990; Vives, 1990; Van Zandt and Vives, 2007).

Definition 3 (Marginal beliefs)**.** An agent of type w has *marginal beliefs* with respect to profile α if she believes that the mass of players in the population following strategy α is $1 - F(w)$, and that the remaining players choose $a = 0$. Accordingly, she believes that the aggregate action is given by $\bar{\alpha}(\alpha, [w, \overline{w}])(1 - F(w))$.

Marginal beliefs have two related properties. First, a player thinks that her type is a pivotal type, i.e., she thinks types lower than hers do not contribute to the aggregate action. Second, she underestimates the value of the aggregate action since $\bar{\alpha}(\alpha, [w, \overline{w}]) \leq \bar{\alpha}(\alpha, [\underline{w}, \overline{w}])$.

A strategy profile α_P maximizes the ex-ante expected payoffs of an agent with marginal beliefs if it satisfies

$$\alpha_P \in \arg\max_{\alpha \in \mathcal{A}} \int_{\underline{w}}^{\overline{w}} U\big(\alpha(w), \bar{\alpha}(\alpha, [w, \overline{w}])(1 - F(w)), \theta, w\big) dF(w). \tag{6}$$

The next result establishes that the set of potential maximizers coincides with the set of profiles that maximize ex-ante payoffs under marginal beliefs. In addition, it shows that potential maximizers exist and coincide with NE strategy profiles except possibly in a set of types with measure zero. To facilitate the exposition, when we say that two strategy profiles coincide *a.e.* (almost everywhere) we mean that the set of types for which they may differ has zero measure. Similarly, we say that a potential maximizer or equilibrium strategy profile is *essentially* unique if any two strategies belonging to the set of maximizers or the equilibrium set differ in a measure zero set of types.

Proposition 2. *If Γ_θ is a weighted potential game then a strategy profile maximizes the (weighted) potential given by* (5) *if and only if it satisfies* (6). *If α^* is a potential maximizer then it coincides a.e. with another potential maximizing strategy profile that is a NE of Γ_θ. In addition, the set of potential maximizers and the set of NE are non-empty.*

The proof of existence relies on showing that, due to the continuity of payoffs and the distribution of types, a potential maximizing strategy profile exists when restricting attention to step functions defined on an interval partition of the type space that assign the same action to all types in the same interval. Then we show that the fact that a player's payoffs only depend on the aggregate action instead of the full strategy profile implies that the potential of any profile can be expressed as the limit potential of a sequence of step functions. The intuition for why a potential maximizer has to coincide with a NE a.e. is linked to the fact that an individual strategy switch makes the players payoff and the potential of the game to move in the same direction. Hence, if there is a set of types that are not best responding, we could switch the actions of an arbitrarily small subset of them so that they best respond, thus increasing potential, without having a noticeable impact on the aggregate action and thus on the contribution to potential of the other players.

Next, we focus on supermodular games satisfying the following assumption.

Assumption 1 (Supermodular game)**.** *Payoffs satisfy*

(i) If $a > a'$ then $\Delta U(a, a', \bar{a}, \theta, w)$ is strictly increasing in $\bar{a}$ and also in w. That is, U exhibits strictly increasing differences w.r.t. a and both $\bar{a}$ and w.

(ii) There exists $K > 0$ such that $U(a, \bar{a}, \theta, w) - U(a', \bar{a}', \theta', w) \geq K(a - a')(\theta - \theta')$ for all $a \geq a'$, all $\bar{a} \geq \bar{a}'$, all $\theta \geq \theta'$ and all w.

Increasing differences with respect to aggregate action in part (i) lead to strategic complementarities. Similarly, part (ii) implies that a higher θ increases the incentives to take higher actions.

It is worth noting that while marginal beliefs do not necessarily reflect how individual incentives to take actions higher than zero compare across types, in supermodular games such beliefs are coherent with the fact that higher types prefer higher actions.

The next results establishes the generic uniqueness of potential maximizers and introduces the function α_P that selects a unique potential-maximizing equilibrium.

Proposition 3. *If Γ is a family of weighted potential games satisfying Assumption 1 then there is an essentially unique strategy profile that maximizes the potential of Γ_θ for all $\theta \in \Theta$ except possibly for a countable subset of Θ. In addition, any potential maximizing strategy profile is increasing a.e. in w.*

Let $\alpha_P(\cdot, \theta)$ be the largest potential-maximizing NE of Γ_θ, with $\alpha_P(w, \theta)$ denoting the strategy of type w. The function $\alpha_P : [\underline{w}, \overline{w}] \times \Theta \to A$ is well-defined and increasing in w and θ.

The equilibrium selection rule α_P selects the essentially unique NE that maximizes potential for almost all θ. In addition, for values of θ at which there is non-trivial multiplicity, i.e., when there are several maximizers that differ in a positive measure set of types, the selection will exhibit discontinuities at θ in some players' strategies associated with the switch to a different equilibrium. The choice of the largest NE is innocuous since only determines the choice of strategy at the zero measure set of θ associated with multiple potential maximizing NE.

The proof of Proposition 3 relies on two properties stemming from complementarities between actions, types and the common parameter θ. First, best responses are increasing in $\bar{a}$ and in w, leading to NE profiles being monotone w.r.t. types. In addition, NE profiles are ordered w.r.t. both individual and aggregate actions. This also means that an agent with marginal beliefs expects the aggregate action to be higher in a higher NE than in a lower NE. Second, since the potential of the game can be expressed as the average payoffs of agents with marginal beliefs, the difference in potential between a higher and a lower NE is increasing in θ given that payoff differences are increasing in θ (Assumption 1). Hence, if there are multiple potential maximizing NE with different aggregate actions at θ then the difference in potential between the highest potential maximizing NE and any lower potential maximizer, which is zero at θ, becomes positive after an infinitesimal increase in θ. By continuity, such a positive difference in potential breaks the tie among potential maximizing NE and yields an essentially unique potential maximizer. Accordingly, if there is non-trivial multiplicity at some θ, it goes away in a neighborhood of θ, implying that the set of θ associated with multiplicity must be at most countable.

We next use the investment game of Example 1 to illustrate how to characterize the potential-maximizing equilibrium selection rule α_P and discuss the economic implications of maximizing expected payoffs under marginal beliefs. Recall that individual payoffs are given by

$$au(\bar{a},\theta) - \frac{a^2}{2}\frac{1}{w},$$

which are strictly concave in a. Hence, the optimal investment for an agent of type w, i.e., the best response to aggregate investment $\bar{a}$, is the solution to the FOC[9]

$$\alpha(w) = u(\bar{a},\theta)w. \tag{7}$$

Integrating individual actions across types and recalling that $Ew = 1$ we obtain the equilibrium condition on aggregate investment:

$$\bar{a} = \int_w u(\bar{a},\theta)w dF(w) = u(\bar{a},\theta). \tag{8}$$

The solutions $\bar{a}^*$ to this equation represent the NE levels of aggregate investment of the game. Given (7) and (8), we can write equilibrium strategies as $\alpha^*(w) = \bar{a}^* w$.

Next, consider the following s-shaped specification of returns: $u(\bar{a},\theta) = 2\theta\dfrac{\bar{a}^2}{\bar{a}^2+1}$. Under s-shaped returns, eq. (8) has at most three solutions, zero investment ($\bar{a}^* = 0$) and the real solutions to quadratic equation

$$\bar{a}^{*2} - 2\theta\bar{a}^* + 1 = 0 \quad \Rightarrow \quad \bar{a}^* = \theta \pm \left(\theta^2 - 1\right)^{1/2}.$$

That is, the game has multiple NE when $\theta \geq 1$. It is worth noting that the NE exhibiting the largest investment Pareto dominate the lower investment equilibria.[10]

[9] To keep things simple we assume that the upper bound a_{max} on investment is high enough so that it is not binding for the values of θ that we consider.

[10] Equilibrium payoffs can be written as $U(\alpha^*(w),\bar{a}^*,\theta,w) = (\bar{a}^* w)\bar{a}^* - \frac{(\bar{a}^* w)^2}{2}\frac{1}{w} = \frac{\bar{a}^*}{2}w$, which are strictly increasing in $\bar{a}^*$.

By Proposition 1 the potential function is given by

$$V(\alpha, \theta) = \int_0^{\bar{\alpha}(\alpha)} u(z, \theta) dz - \int_w \frac{\alpha(w)^2}{2} \frac{1}{w} dF(w). \tag{9}$$

Since NE satisfy $\alpha^*(w) = \bar{a}^* w$ and strategy profiles that maximize potential are essentially equal to a NE, we can restrict attention to individual strategies of the form $\alpha_{\bar{a}}(w) = \bar{a}w$ and look for the set of $\bar{a}$ that maximize

$$V(\alpha_{\bar{a}}, \theta) = \int_0^{\bar{a}} 2\theta \frac{z^2}{z^2+1} dz - \int_w \frac{\bar{a}^2 w^2}{2} \frac{1}{w} dF(w) = 2\theta \left(\bar{a} - \tan^{-1}(\bar{a})\right) - \frac{\bar{a}^2}{2}. \tag{10}$$

The left plot in Figure 1 shows that this function has at most two local maximizers, the zero investment equilibrium and the largest solution to eq. (8) whenever it exists.[11] As the quality of the technology crosses threshold $\hat{\theta} \approx 1.1$, the global maximizer, denoted by $\bar{a}_P$, switches. This implies that individual investment strategies remain at zero at quality levels below the threshold and discontinuously jump at $\hat{\theta}$, as depicted in the right graph of Figure 1. Specifically, they are given by the cutoff function

$$\alpha_P(w, \theta) = \begin{cases} 0 & \theta < \hat{\theta} \\ \left(\theta + (\theta^2 - 1)^{1/2}\right) w & \theta \geq \hat{\theta}. \end{cases} \tag{11}$$

The zero-investment trap and the discontinuous switch to the high investment

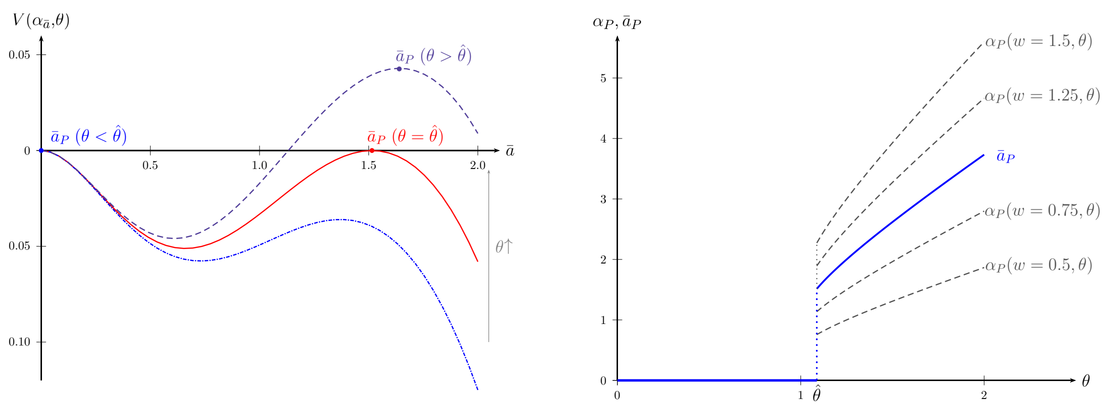

Figure 1: Potential maximization (left) and investment strategies (right).

[11]The NE with intermediate investment levels is a local minimizer.

equilibrium is due to s-shaped returns, which imply that complementarities are small at low and high investment levels, while being stronger at moderate values of $\bar{a}$. Maximizing the ex-ante payoffs of an agent with marginal beliefs leads to selecting the no investment equilibrium over the Pareto dominant NE for all $\theta \in [1, \hat{\theta})$. This is driven by the fact that marginal beliefs underestimate investment levels, making the largest investment equilibrium look too risky from an ex-ante perspective when the quality of the technology θ is not very high.

4.2 Global Games Selection

We next describe the Global Games (GG) equilibrium selection rule for supermodular games (Carlsson and van Damme, 1993; Frankel et al., 2003). It is based on introducing incomplete information about parameter θ via idiosyncratic noise. Specifically, each agent gets a signal $s = \theta + \nu\eta$, where $\nu > 0$ is the noise scale, and η is independently drawn from a continuous distribution with full support on $[-1/2, 1/2]$. The noise distribution is allowed to depend on the agent's type. Let H_w and h_w respectively denote the cdf and the density of ν for an agent of type w. Agents have a common prior about θ with continuous density ϕ and full support on Θ. We assume that the exact Law of Large Numbers (LLN) applies for each type w, i.e., the fraction of agents of type w with signal noise less than ε is given by $H_w(\varepsilon)$. A strategy in the incomplete information game is a mapping from the space of signals S to actions in A. Abusing notation, let $\alpha : [\underline{w}, \overline{w}] \times S \to A$ denote the strategy profile that assigns action $\alpha(w, s)$ to a player of type w receiving signal s.

The goal of the GG selection is to induce uniqueness of Bayes Nash equilibrium (BNE) in the incomplete information game and then select a NE of the complete information game Γ_θ by taking the limit as $\nu \to 0$. To obtain uniqueness, in addition to Assumption 1, we introduce dominance regions, that is, ranges of parameter values at which all player types have a strictly dominant strategy.

Assumption 2. *There exist $\underline{\theta} > \inf \Theta$ and $\bar{\theta} < \sup \Theta$ such that, for all w and all $\bar{a} \in [0, a_{max}]$, if $\theta < \underline{\theta}$ then $\Delta U(a, 0, \bar{a}, \theta, w) < 0$ for all $a > 0$, and if $\theta > \bar{\theta}$ then $\Delta U(a_{max}, a, \bar{a}, \theta, w) > 0$ for all $a < a_{max}$.*

The next two propositions establish, respectively, that there is an essentially unique BNE (i.e., unique except in a zero measure subset of $[\underline{w}, \overline{w}] \times S$), and that it converges to a NE of Γ_θ as noise vanishes.

Proposition 4. *If Assumptions 1 and 2 are satisfied then there exists $\bar{\nu} > 0$ such that for all $\nu < \bar{\nu}$ there is an essentially unique BNE in the global game. Moreover, the equilibrium strategy profile is monotone in both s and w.*

The proof is based on standard arguments in the global games literature (Frankel et al., 2003). First, payoffs exhibiting increasing differences w.r.t. signals and actions implies that the global game is a supermodular game. Thus, the game has both a smallest and a largest equilibrium (Theorem 5 in Milgrom and Roberts, 1990). Moreover, players follow monotone strategies in these equilibria. Second, we show that, under a uniform prior, shifting all strategies so that the shifted action under signal $s+\delta$ correspond to the original action under signal s leads to the same beliefs about the aggregate action at the shifted signal $s+\delta$, but to higher expectations about θ. We exploit this translation invariance to prove that, as we move up from the smallest to the largest equilibrium, expected payoffs differences between higher and lower actions go up. This implies that there can be only one equilibrium modulo zero measure sets of types and signals. Finally, we show that beliefs under a non-uniform prior converge to the beliefs associated with a uniform prior.

Proposition 5. *If Assumptions 1 and 2 are satisfied then there exists a strategy profile α_G such that, (i) for any sequence α^ν of equilibria in the global game indexed by $\nu \to 0$, $\lim_{\nu \to 0} \alpha^\nu(w, s) = \alpha_G(w, s)$ for almost all w, s; and (ii) $\alpha_G(\cdot, \theta)$ is a NE of Γ_θ for almost all $\theta \in \Theta$.*

A desirable property of the GG selection is to be invariant to different noise distributions H_w. In such a case, the GG selection is said to be *noise-independent*. As we show next, this is indeed the case under quasilinear payoffs, given its equivalence to potential maximization.

4.3 Equivalence Result

Theorem 1 establishes below that the limit equilibrium α_G in the global game essentially coincides with the potential maximizing strategy profile α_P in the complete information game. We obtain the result by identifying a key property of beliefs about the aggregate action in the global game, which we call the *Generalized Laplacian Property* (GLP). The GLP links the (weighted) average belief to the uniform distribution. We use the GLP to show that, under quasilinear payoffs, the change in expected payoffs of a type that switches actions coincides with the change in potential as noise vanishes. Since this is a key step in the proof, we use Example 1 to introduce the GLP and provide some intuition before formally stating the theorem.

Recall that, in the investment game, the potential maximizing profile $\alpha_P(w, \theta)$ given by (11) implies that firms follow a cutoff rule, i.e., they choose zero investment if $\theta < \hat{\theta}$ and the largest NE investment if $\theta \geq \hat{\theta}$ (see Figure 1). Accordingly, for the limit equilibrium in the global game to coincide with $\alpha_P(w, \theta)$, a firm with type w needs to be indifferent between choosing $\alpha_1(w) = 0$ and $\alpha_2(w) = \alpha_P(w, \hat{\theta})$ when it receives signal $s = \hat{\theta}$. That is, the expected payoff difference between the two investment levels conditional on $s = \hat{\theta}$ must be zero and therefore, the average across types of such payoff differences must also be zero. Using the GLP we will show that the average of expected payoff differences is indeed equal to the difference in potential between strategy profiles α_1 and α_2, which is zero since both profiles maximize potential at $\hat{\theta}$.

Assume that in the global game a firm of type w chooses $\alpha_1(w)$ if its signal is below some cutoff $\kappa(w)$ and $\alpha_2(w)$ if its signal is above $\kappa(w)$. Furthermore, assume that firms have a uniform prior over Θ and that their cutoffs $\kappa(\cdot)$ are within noise range ν of each other. Notice that, when ν is very small, θ is very close to the value of the firm's signal, and expected payoffs conditional on signal $s = \kappa(w)$ can be approximated by

$$E[U(a, \bar{a}, \theta, w)|s = \kappa(w)] \approx \int_z au(z, \kappa(w))dG_w(z) - \frac{a^2}{2}\frac{1}{w},$$

where $G_w(z) = Pr(\bar{a} \leq z|s = \kappa(w), w)$ denotes the cdf of aggregate investment conditional on signal $s = \kappa(w)$ and on type w when players follow the above cutoff strategies. Hence, the difference in expected payoffs across the two actions can be expressed as

$$0 \approx \int_z (\alpha_2(w) - \alpha_1(w))u(z, \kappa(w))dG_w(z) - \frac{\alpha_2(w)^2 - \alpha_1(w)^2}{2}\frac{1}{w}.$$

Solving these indifference conditions requires pinning down G_w, which depends on the distribution of noise and signal cutoffs κ. However, under quasilinear payoffs and a uniform prior, we can circumvent this problem by focusing on *average* instead of individual indifference conditions and using the GLP to replace average beliefs.

The next lemma presents the GLP under a uniform prior, which states that the weighted average of beliefs G_w is given by the uniform distribution. The weights represent the contribution of type w to the aggregate action when it switches actions, i.e., they are given by $(\alpha_2(w) - \alpha_1(w))f(w)$.

Lemma 1 (Generalized Laplacian Property)**.** *Fix $\alpha_1, \alpha_2 \in \mathcal{A}$ satisfying $\alpha_1(w) \leq \alpha_2(w)$ for all w. If players have a uniform prior then, for any signal cutoff function $\kappa : [\underline{w}, \bar{w}] \to [\inf \Theta + \nu/2, \sup \Theta - \nu/2]$ such that each type w chooses $\alpha_1(w)$ if $s < \kappa(w)$ and $\alpha_2(w)$ if $s \geq \kappa(w)$ and for all $z \in [\bar{\alpha}(\alpha_1), \bar{\alpha}(\alpha_2)]$, we have that*[12]

$$\int_{\underline{w}}^{\overline{w}} (\alpha_2(w) - \alpha_1(w)) G_w(z) dF(w) = z - \bar{\alpha}(\alpha_1). \tag{12}$$

Averaging indifference conditions across firms we obtain

$$0 \approx \int_w \int_z (\alpha_2(w) - \alpha_1(w)) u(z, \kappa(w)) dG_w(z) dF(w) - \int_w \frac{\alpha_2(w)^2 - \alpha_1(w)^2}{2} \frac{1}{w} dF(w).$$

In addition, as ν goes to zero, since firms' signal cutoffs are within the noise range of each other, $u(z, \kappa(w))$ can be approximated by $u(z, k)$, where k is the limit cutoff to which all $\kappa(w)$ converge. Hence, the first integral in the above expression satisfies

$$\int_z u(z,k) d\Big(\int_w (\alpha_2(w) - \alpha_1(w)) G_w(z) dF(w)\Big) = \int_{\bar{\alpha}(\alpha_1)}^{\bar{\alpha}(\alpha_2)} u(z,k) dz,$$

where the equality comes from applying (12) to replace the average belief while the limits of integration reflect the range of feasible aggregate investment levels given the cutoff strategies used by firms. This implies that cutoff k must satisfy

$$0 \approx \int_{\bar{\alpha}(\alpha_1)}^{\bar{\alpha}(\alpha_2)} u(z,k) dz - \int_w \frac{\alpha_2(w)^2 - \alpha_1(w)^2}{2} \frac{1}{w} dF(w).$$

But notice that, when $k = \hat{\theta}$, the RHS is the difference in potential between NE profiles α_2 and α_1 at quality $\hat{\theta}$, which is zero since both profiles maximize potential.

The relationship between expected payoff differences and changes in potential

[12] To see how (12) implies that $\bar{a}$ is uniformly distributed in $[\bar{\alpha}(\alpha_1), \bar{\alpha}(\alpha_2)]$ note that we can divide both sides by $\bar{\alpha}(\alpha_2) - \bar{\alpha}(\alpha_1)$ so that the RHS is equal to uniform cdf $\frac{z - \bar{\alpha}(\alpha_1)}{\bar{\alpha}(\alpha_2) - \bar{\alpha}(\alpha_1)}$.

partially illustrated by this example can be established for each type individually by applying the GLP to any subset of types, effectively linking changes in expected payoffs following a deviation from the potential maximizing equilibrium to changes in potential. This leads to the following equivalence result.

Theorem 1. *If Γ is a family of weighted potential games and satisfies Assumptions 1 and 2 for all $\theta \in \Theta$ then, for any prior with continuous density and full support in Θ, α_G is equal to α_P, except possibly in a measure zero subset of $[\underline{w}, \overline{w}] \times \Theta$. Accordingly, α_G is noise-independent.*

The equivalence result provides a tractable characterization of the GG selection as well as an economic interpretation in terms of maximizing ex-ante payoffs under marginal beliefs. In addition, it identifies quasilinearity as a sufficient, easy-to-check condition for noise independence in aggregative global games.

The proof works as follows. It first shows for the uniform-prior case that, for any subset of types W, the expected average payoff gain across types in W from deviating from α_P to any profile α using a common signal threshold $s = \theta$ is equal to the change in potential. This is done by applying a version of the GLP showing that the average belief about the aggregate action of types in W is the uniform distribution (Lemma 3 in Appendix A). Since we can choose W to be a small neighborhood of any type w and potential maximization implies that the change in potential must be negative, a continuity argument implies that, for almost all types, the deviation from α_P to α is not profitable, i.e., that the potential maximizing equilibrium profile coincides with the limit equilibrium in the global game. The proof then uses a limit version of the GLP for non-uniform priors to extend the equivalence result beyond the uniform-prior case (Lemma 5 in Appendix A).[13]

The GLP holds irrespective of the payoff structure of the game and is the product of combining three key ingredients, namely, monotone cutoff strategies, a uniform prior and additive noise. Cutoff strategies link the aggregate action of type w to the

[13]The limit version of the GLP could be used to relax the assumption of additive noise. For instance, any signals of the form $s = d(\theta, \eta; \nu)$, with d strictly increasing, for which there exists a monotone transformation $q(s) = q_1(\theta) + \nu q_2(\eta)$ would lead to the same limit equilibrium. These include multiplicative noise, i.e., $s = \theta\eta^{\nu}$, as well as exponential noise ($s = \theta^{\eta^{\nu}}$), with noise support defined to ensure that signals are well-defined and monotone in θ and η. The reason is that we can redefine signals as $q(s)$, the common value parameter as $q_1(\theta)$ and noise as $q_2(\eta)$ so that the associated distributions satisfy all the assumptions, thus leading to the same equilibrium selection.

proportion of agents with signals higher than the type's cutoff, i.e., to the cdf of signals. In turn, the uniform prior and additive noise lead to signals being uniformly distributed and independent of types, as long as signals lie in $[\inf \Theta + \nu/2, \sup \Theta - \nu/2]$. This is because, as shown by Lemma 4 in Appendix A, the sum of two independent random variables, one of them uniformly distributed and with a larger support than the other, has a constant density except at the tails. This explains both the connection between average beliefs and the uniform distribution and also their invariance with respect to H_w.

The above example also illustrates the key role that quasilinearity plays in the characterization of the global games selection and, particularly, in its independence of the noise structure. If the payoff impact of the aggregate action is not linear in own action or it is asymmetric across types then we cannot apply the GLP to replace weighted average beliefs. This is because in the former case beliefs weights are not proportional to the difference in actions $(\alpha_2(w) - \alpha_1(w))$, while in the latter case we cannot integrate individual beliefs separately from payoffs if u were to vary across types. Hence, solving for equilibrium would require the use of individual beliefs, which depend in non-trivial ways on the distribution of noise (see Serrano-Padial (2018) for an example in which the GG selection depends on H_w when payoffs are not quasilinear).

5 Comparative Statics

This section identifies conditions under which changes in parameters (θ) or heterogeneity (F) makes the set of NE aggregate actions to go up, even if the game is not supermodular. By exploiting the quasilinear payoff structure, we are able to provide conditions on average payoffs. To do so, we first characterize the set of NE aggregate actions as fixed points of a nested maximization problem.

Let $\mathcal{A}_{\bar{a}} = \{\alpha \in \mathcal{A} : \bar{\alpha}(\alpha) = \bar{a}\}$ be the set of strategy profiles with aggregate action $\bar{a}$. In addition, denote by $B(\bar{a}, \theta)$ the maximum average value of the idiosyncratic payoff component $v(a, \theta, w)$ when strategy profiles are restricted to $\mathcal{A}_{\bar{a}}$, that is,

$$B(\bar{a}, \theta) := \max_{\alpha \in \mathcal{A}_{\bar{a}}} \int_w v(\alpha(w), \theta, w) dF(w). \tag{13}$$

Lemma 2. *$B(\bar{a},\theta)$ is well-defined and continuous in $\bar{a}$. If payoffs are quasilinear then $\bar{a}^*$ is the aggregate action in some NE if and only if*

$$\bar{a}^* \in \underset{\bar{a}\in[0,a_{max}]}{\arg\max} \left(\bar{a}u(\bar{a}^*,\theta) + B(\bar{a},\theta)\right). \tag{14}$$

In words, the aggregate action $\bar{a}^*$ in a NE is the solution to a two-step problem. First, find the highest average idiosyncratic payoffs associated with each level of the aggregate action $\bar{a}$. Second, check if aggregate action $\bar{a}^*$ maximizes average total payoffs among all possible aggregate actions $\bar{a}$ when the symmetric payoff component is kept fixed at $u(\bar{a}^*,\theta)$.

This result allows us to establish conditions on $u(\bar{a},\theta)$ and $B(\bar{a},\theta)$ for the set of NE aggregate actions to go up after an increase in θ (Theorems 2 and 3) or after a change in the type distribution F (Theorems 4 and 5).

Theorem 2 (Robust Comparative Statics I)**.** *Let payoffs be quasilinear and let Θ' be a compact subset of Θ. If $u(\bar{a},\theta)$ is increasing in $\theta\in\Theta'$ for all $\bar{a}\in[0,a_{max}]$ and $B(\bar{a},\theta)$ exhibits strictly increasing differences in $[0,a_{max}]\times\Theta'$ then the smallest and largest NE aggregate actions are increasing in Θ'.*

Theorem 2 can be further generalized by assuming that $\bar{a}u(\bar{a}',\theta)+B(\bar{a},\theta)$ is single-crossing in $\bar{a},\theta$ for all $\bar{a}'$ (Milgrom and Shannon, 1994). However, imposing separate conditions on u and B allows for a simpler derivation of direct restrictions on payoffs. Still, the conditions on $B(\bar{a},\theta)$ can be hard to interpret since they involve the value function of a constrained maximization problem. The next result provides sufficient conditions for the smallest and a largest NE to go up with θ in the continuous-action case. Let $int(X)$ denote the interior of a set X. In particular, $int(\mathcal{A})$ is the set of measurable profiles such that $\alpha(w)\in(0,a_{max})$ for all w.

Theorem 3. *Let $A=[0,a_{max}]$ and $\Theta'\subseteq\Theta$ be a closed interval of parameter values θ. Let $u(\bar{a},\theta)$ be increasing in $\theta\in\Theta'$ for all $\bar{a}\in[0,a_{max}]$. If the smallest and a largest NE are in $int(\mathcal{A})$ for all $\theta\in\Theta'$ and, for any $\alpha\in int(\mathcal{A})$ and any $\theta\in int(\Theta')$,*

$$\int_{\underline{w}}^{\overline{w}} \frac{\partial^2 v(a,\theta,w)}{\partial a\partial\theta}\Big|_{a=\alpha(w)} dF(w) > 0, \tag{15}$$

then the smallest and a largest NE aggregate actions are increasing in Θ'.

Condition (15) is straightforward to check and it requires the idiosyncratic payoff component to exhibit *average* increasing differences in $int(\mathcal{A}) \times int(\Theta')$. For instance, in the negative externalities game of Example 2 this amounts to the expected marginal benefit from consumption being increasing in θ. To see why, recall that in the example payoffs are given by

$$U(a, \bar{a}, \theta, w) = b(a, \theta, w) - c(\bar{a}, \theta)a,$$

where b is strictly concave in a and c is increasing in $\bar{a}$. Concavity of b implies that utility is strictly concave in a. Hence, as long as the marginal benefit satisfies $\frac{\partial b(a,\theta,w)}{\partial a}\Big|_{a=0} > c(\bar{a}, \theta)$ and a_{max} is large enough, the optimal consumption levels will lie in the interior of $[0, a_{max}]$. Hence, condition (15) translates into

$$\int_{\underline{w}}^{\overline{w}} \frac{\partial^2 b(a, \theta, w)}{\partial a \partial \theta}\Big|_{a=\alpha(w)} dF(w) = \frac{\partial}{\partial \theta} E\left(\frac{\partial b(a, \theta, w)}{\partial a}\Big|_{a=\alpha(w)}\right) > 0.$$

Condition (15) is significantly weaker than imposing monotonicity restrictions pointwise at the type level (Milgrom and Shannon, 1994; Acemoglu and Jensen, 2010). For instance, it is straightforward to check that if payoffs exhibit increasing differences for all types then they satisfy the conditions in Theorem 3. In contrast, Theorem 3 allows for individual payoffs of a subset of types to exhibit strict *decreasing* differences in a, θ.[14] In addition, the theorem presents conditions on payoffs, which are primitives of the game, making them easier to check than restrictions on average best responses (Camacho et al., 2018).

Theorem 4 (Robust Comparative Statics II)**.** *Let $B(\bar{a}, \theta)$ and $\hat{B}(\bar{a}, \theta)$ be the value functions defined by* (13) *associated with type distributions F and $\hat{F}$, respectively. If*

$$B(\bar{a}, \theta) - B(\bar{a}', \theta) \geq \hat{B}(\bar{a}, \theta) - \hat{B}(\bar{a}', \theta)$$

for any $\bar{a}, \bar{a}' \in [0, a_{max}]$ such that $\bar{a} > \bar{a}'$, then the smallest and largest NE aggregate actions are higher under F than under $\hat{F}$.

The next theorem presents sufficient conditions that are easy to verify for games with continuous actions.

[14] In Example 2 payoffs U exhibit decreasing differences if $\frac{\partial^2 U}{\partial a \partial \theta} \leq 0$, that is, if $\frac{\partial^2 b}{\partial a \partial \theta} < \frac{\partial c}{\partial \theta}$.

Theorem 5. *Let $A = [0, a_{max}]$ and $F, \hat{F}$ be two type distributions. If the smallest and largest NE are in $int(\mathcal{A})$ for both F and $\hat{F}$, and, for any $\alpha \in int(\mathcal{A})$,*

$$\int_{\underline{w}}^{\overline{w}} \frac{\partial v(a,\theta,w)}{\partial a}\Big|_{a=\alpha(w)} dF(w) > \int_{\underline{w}}^{\overline{w}} \frac{\partial v(a,\theta,w)}{\partial a}\Big|_{a=\alpha(w)} d\hat{F}(w), \tag{16}$$

then the smallest and largest NE aggregate actions are higher under F than under $\hat{F}$.

In Example 2 condition (16) boils down to

$$\int_{\underline{w}}^{\overline{w}} \frac{\partial b(a,\theta,w)}{\partial a}\Big|_{a=\alpha(w)} dF(w) \geq \int_{\underline{w}}^{\overline{w}} \frac{\partial b(a,\theta,w)}{\partial a}\Big|_{a=\alpha(w)} d\hat{F}(w).$$

This is the case, for instance, if $\frac{\partial b}{\partial a}$ is increasing in w and F first-order stochastically dominates $\hat{F}$, or if $\frac{\partial b}{\partial a}$ is concave in w and $\hat{F}$ is a mean-preserving spread of F.

We finish by mentioning that, although beyond the scope of the paper, the above characterization of the set of NE aggregate actions (Lemma 2) could be used to explore whether there exist average conditions on payoffs that guarantee uniqueness of the equilibrium aggregate action, instead of relying on pointwise conditions.[15]

6 Related Literature

Our discussion of the related literature narrowly focuses on the three more closely related areas, namely, large potential games, heterogeneous global games, and comparative statics in aggregative games.

Monderer and Shapley (1996) introduced various definitions of potential in finite games. Among other properties, potential maximizers have been shown to be evolutionary stable. In the context of large games, Sandholm (2009) defined potential in population games with finite actions and finite types, and showed that the existence of potential in these games is linked to payoffs exhibiting externality symmetry.[16] Our notion of potential extends the definition of potential to allow

[15] Cheung and Lahkar (2018) and Lahkar (2017) study equilibrium existence in potential aggregative games with a homogeneous population.

[16] Hofbauer and Sandholm (2007) and Zusai (2018) prove the evolutionary stability of local maximizers of potential in population games with finite actions and player types.

for continuous actions and types, and we show why the existence of potential implies quasilinear payoffs by deriving symmetry conditions for payoffs that depend on the aggregate action. We also identify the potential function for this class of aggregative games, leading to a tractable characterization of potential maximizing equilibria, and provide economic content behind potential maximization, which has been an open question in the literature. Our extension to continuous type games also builds on the literature on non-atomic games (Schmeidler, 1973).

Frankel et al. (2003) proposed the global games selection for games with heterogeneous payoffs and established uniqueness of the selected equilibrium in both finite- and continuum-player games with finite types. Focusing on binary-action games with symmetric NE, Sakovics and Steiner (2012) identified a key property of average beliefs that they used to characterize the global games selection. Drozd and Serrano-Padial (2018) extended such characterization to binary action games with asymmetric equilibria. Our results build upon their work by generalizing the belief property beyond binary actions and by characterizing the global games selection using the potential function. The connection between potential maximization and the global games selection can be established for finite supermodular games by combining the results of Ui (2001) and Morris and Ui (2005), who respectively show that maximizers of potential and the more general notion of monotone potential are robust to incomplete information in the sense of Kajii and Morris (1997),[17] with those of Basteck et al. (2013), who show that the global games selection picks the robust equilibrium whenever it exists. In addition to directly proving this connection in large aggregative games, since the potential maximizing equilibrium is unrelated to the distributional assumptions about noise used in the global game, we provide an easy-to-check sufficient condition, quasilinearity, for the global games selection to be noise independent.

Finally, the paper contributes to the recent literature of aggregate comparative statics (Acemoglu and Jensen, 2010, 2015; Camacho et al., 2018), which focuses on comparative statics on aggregate behavior instead of on individual choices (Topkis, 1979; Milgrom and Roberts, 1990; Vives, 1990; Milgrom and Shannon, 1994). Acemoglu and Jensen (2010, 2015) find monotonicity conditions on individual best

[17]Potential maximization implies monotone potential maximization. Oyama and Takahashi (2020) prove that monotone potential maximization is necessary and sufficient for robustness in binary-action finite games.

responses for the smallest and largest equilibrium aggregate actions to be monotone in the model parameters. Camacho et al. (2018) further relax these restrictions by pinning down monotonicity conditions on average best responses. We expand their contributions by identifying direct restrictions on average payoffs.

Appendix A The Generalized Laplacian Property

This section presents the full version of the GLP for uniform prior and the limit version of the GLP for non-uniform priors.

Lemma 3 (Generalized Laplacian Property)**.** *Assume agents have a uniform prior. Fix any measurable subset of types W, any $\alpha_1, \alpha_2 \in \mathcal{A}$ with $\alpha_1(w) \leq \alpha_2(w)$ for all $w \in W$, and any cutoff function $\kappa : [\underline{w}, \bar{w}] \to [\inf \Theta + \nu/2, \sup \Theta - \nu/2]$ such that $\alpha(w,s) = \alpha_1(w)$ if $s < \kappa(w)$ and $\alpha(w,s) = \alpha_2(w)$ if $s \geq \kappa(w)$ for all $w \in W$. Then, for all $z \in [\bar{\alpha}(\alpha_1, W), \bar{\alpha}(\alpha_2, W)]$,*

$$\int_W (\alpha_2(w) - \alpha_1(w)) G_w(z|\kappa; \alpha, W) dF(w|w \in W) = z - \bar{\alpha}(\alpha_1, W), \tag{17}$$

where $G_w(z|\kappa;\alpha,W) := Pr(\bar{\alpha}(\alpha,W) < z)|s = \kappa(w), w)$.

To prove Lemma 3 we make use of the following property of the sum of two independent random variables.

Lemma 4. *Let x, y be two independent random variables such that x is uniformly distributed in $[\underline{x}, \overline{x}]$ and y has a density f_y with support $[\underline{y}, \overline{y}]$. If $\overline{y} - \underline{y} < \overline{x} - \underline{x}$ then the sum $z = x + y$ has a constant density in $[\underline{x} + \overline{y}, \overline{x} + \underline{y}]$. Specifically,*

$$f_z(z) = \begin{cases} \frac{1}{\overline{x}-\underline{x}} F_y(z - \underline{x}) & z < \underline{x} + \overline{y} \\ \frac{1}{\overline{x}-\underline{x}} & z \in [\underline{x} + \overline{y}, \overline{x} + \underline{y}] \\ \frac{1}{\overline{x}-\underline{x}} \left(1 - F_y(z - \overline{x})\right) & z > \overline{x} + \underline{y}. \end{cases} \tag{18}$$

Proof of Lemma 4. Note that, since $x = z - y$ for any given z, we must have that $x \in [\max\{\underline{x}, z - \overline{y}\}, \min\{\overline{x}, z - \underline{y}\}]$. The joint density of z and x is given by $f_z(z|x) f_x(x)$. In addition, $f_z(z|x) = f_y(z - x|x) = f_y(z - x)$ since y is independent of x. Hence, the density of z satisfies $f_z(z) = \int_{\underline{x}}^{\overline{x}} f_z(z|x) f_x(x) dx = \int_{\max\{\underline{x}, z-\overline{y}\}}^{\min\{\overline{x}, z-\underline{y}\}} f_y(z-x) \frac{1}{\overline{x}-\underline{x}} dx$, leading to the following expression:

$$f_z(z) = \frac{1}{\overline{x} - \underline{x}} \left(F_y(z - \max\{\underline{x}, z - \overline{y}\}) - F_y(z - \min\{\overline{x}, z - \underline{y}\}) \right),$$

which yields (18) by plugging the values of $\max\{\underline{x}, z - \overline{y}\}$ and $\min\{\overline{x}, z - \underline{y}\}$ for the following three cases. First, if $z < \underline{x} + \overline{y}$ then $\max\{\underline{x}, z - \overline{y}\} = \underline{x}$ and $\min\{\overline{x}, z - \underline{y}\} = z - \underline{y}$. Second, if $z \in [\underline{x} + \overline{y}, \overline{x} + \underline{y}]$ then $\max\{\underline{x}, z - \overline{y}\} = z - \overline{y}$ and $\min\{\overline{x}, z - \underline{y}\} = z - \underline{y}$. Finally, if $z > \overline{x} + \underline{y}$ then $\max\{\underline{x}, z - \overline{y}\} = z - \overline{y}$ and $\min\{\overline{x}, z - \underline{y}\} = \overline{x}$. □

Proof of Lemma 3. The proof consists of two parts. The first shows that when agents follow cutoff strategy κ the aggregate action coincides with the aggregate action in a game where the set of available actions is normalized to be $\{0,1\}$ and the type distribution is weighted by the difference $\alpha_2(w)-\alpha_1(w)$. The second part shows that, in the normalized game, the average belief conditional on $s=k(w)$ about the aggregate action of types in W is uniformly distributed in $[0,1]$.

Abusing notation, for any $\theta \in \Theta$, let $\bar{\alpha}(\alpha, W, \theta)$ denote the aggregate action under the strategy $\alpha(w,s)=\alpha_1(w)$ if $s<\kappa(w)$ and $\alpha(w,s)=\alpha_2(w)$ otherwise. By the exact LLN, the fraction of agents of type w that receive a signal below cutoff $\kappa(w)$ is given by $1-H_w\left(\frac{k(w)-\theta}{\nu}\right)$. Accordingly, the aggregate action associated with types in W when they follow cutoff strategy κ is given by

$$\bar{\alpha}(\alpha, W, \theta)=\bar{\alpha}(\alpha_1, W)+\int_W (\alpha_2(w)-\alpha_1(w))\left(1-H_w\left(\frac{k(w)-\theta}{\nu}\right)\right) f(w|w\in W)dw.$$

Define the density function $\hat{f}(w|w\in W)=\frac{\alpha_2(w)-\alpha_1(w)}{\bar{\alpha}(\alpha_2,W)-\bar{\alpha}(\alpha_1,W)}f(w|w\in W)$. Note that $\hat{f}$ is well-defined since $\bar{\alpha}(\alpha_i,W)=\displaystyle\int_W \alpha_i(w)f(w|w\in W)dw$ for $i=1,2$. Let $\hat{F}$ be the corresponding cdf.

We can express the aggregate action in W as

$$\bar{\alpha}(\alpha, W, \theta)=\bar{\alpha}(\alpha_1, W)+(\bar{\alpha}(\alpha_2, W)-\bar{\alpha}(\alpha_1, W))y(\kappa, W, \theta), \tag{19}$$

where $y(\kappa, W, \theta)=\int_W \left(1-H_w\left(\frac{k(w)-\theta}{\nu}\right)\right)\hat{f}(w|w\in W)dw$ represents the mass of agents in W with signals $s\geq\kappa(w)$. Accordingly, we have that

$$\begin{aligned}&\int_W (\alpha_2(w)-\alpha_1(w))G_w(z|\kappa;\alpha,W)dF(w|w\in W)\\ &=\int_W Pr\left(y(\kappa,W,\theta)<\frac{z-\bar{\alpha}(\alpha_1,W)}{\bar{\alpha}(\alpha_2,W)-\bar{\alpha}(\alpha_1,W)}\Big|s=\kappa(w),w\right)d\hat{F}(w|w\in W) \quad (20)\end{aligned}$$

Since $\frac{z-\bar{\alpha}(\alpha_1,W)}{\bar{\alpha}(\alpha_2,W)-\bar{\alpha}(\alpha_1,W)}\in[0,1]$ when $z\in[\bar{\alpha}(\alpha_1,W),\bar{\alpha}(\alpha_2,W)]$, to prove (17) it suffices to show that

$$\int_W Pr\left(y(\kappa,W,\theta)<z|s=\kappa(w),w\right)d\hat{F}(w|w\in W)=z \text{ for all } z\in[0,1]. \tag{21}$$

To do so, consider the normalized game $\hat{\Gamma}_\theta=\{\hat{F},\{0,1\},\theta,U\}$. If agents in the global game version of $\hat{\Gamma}_\theta$ follow strategy $\alpha(w,s)=0$ if $s<\kappa(w)$ and $\alpha(w,s)=1$ otherwise, then the aggregate action in W is given by $y(\kappa,W,\theta)$ for all θ.

To prove (21), define 'virtual signals' $\tilde{s}=s-\kappa(w)$ for all $w\in W$, which exhibit a common cutoff $\tilde{\kappa}=0$. Let the 'extended type' of a player be the tuple (s,w).

Since θ is uniformly distributed in $[\inf\Theta,\sup\Theta]$ and $\nu\eta$ is independent of θ with support $[-\nu/2,\nu/2]$, by Lemma 4, signals in $[\inf\Theta+\nu/2,\sup\Theta-\nu/2]$ have constant density $\frac{1}{\sup\Theta - \inf\Theta}$ independent of w. Accordingly, the density associated with extended type $(k(w), w)$, conditional on $\tilde{s} = 0$ and on $w \in W$, is given by

$$Pr(\kappa(w), w|\tilde{s} = 0, W) = \frac{Pr(k(w), w|W)}{Pr(\tilde{s} = 0|W)} = \frac{\frac{1}{\sup\Theta - \inf\Theta}\hat{f}(w|W)}{\frac{1}{\sup\Theta - \inf\Theta}} = \frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}. \quad (22)$$

where $Pr(s, w|\cdot)$ denotes the conditional probability density of extended type (s, w).

Next, we show that $y(\kappa, W, \theta)$ is uniformly distributed conditional on $\tilde{s} = 0$, i.e.,

$$Pr(y(\kappa, W, \theta) < z|\tilde{s} = 0, W) = z. \quad (23)$$

First note that the virtual noise $\tilde{\eta} = (\tilde{s} - \theta)/\nu$ follows the mixture distribution $\left\{H_w\left(\tilde{\eta} + \frac{\kappa(w)}{\nu}\right), \frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}\right\}_W$. This implies that the virtual noise belongs to type w with probability $\frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}$. In addition, its distribution conditional on type w is given by the noise distribution evaluated at $\eta = \tilde{\eta} + \kappa(w)/\nu$. But note that the mixture distribution does not depend on θ so the random variable $\tilde{\eta}$ is i.i.d. across agents and independent of θ.

Let $\hat{H}$ be the cdf of $\tilde{\eta}$ and define $\hat{H}^{-1}(z) = \inf\{\tilde{\eta} : \hat{H}(\tilde{\eta}) = z\}$. Given the definition of virtual noise, the aggregate action in subset W is given by the fraction of agents in W whose virtual signal is greater than zero, i.e., by one minus the cdf of the virtual noise $\hat{H}$ evaluated at $-\theta/\nu$. This yields expression (23) given that

$$\begin{aligned} Pr(y(\kappa, W, \theta) < z|\tilde{s} = 0, W) &= Pr(1 - \hat{H}(-\theta/\nu) < z|\tilde{s} = 0, W) = Pr(1 - \hat{H}(\tilde{\eta}) < z) \\ &= Pr(\tilde{\eta} > \hat{H}^{-1}(1 - z)) = 1 - \hat{H}(\hat{H}^{-1}(1 - z)) = z. \end{aligned}$$

Combining (22) and (23) we obtain (21), since

$$Pr(y(\cdot) < z|\tilde{s} = 0, W) = \int Pr(y(\cdot) < z|s = \kappa(w), w)Pr(s = \kappa(w), w|\tilde{s} = 0, W)dw. \quad \square$$

Lemma 4 reveals the key role that the uniform prior and additive noise play by inducing a uniform distribution of signals. Nonetheless, a version of the GLP for non-uniform priors approximately holds when noise levels are very small, in which the weights on individual beliefs depend on the prior.

Lemma 5 (Generalized Laplacian Property for Non-uniform Prior)**.** *Assume agents have a common prior with continuous density ϕ that has full support on Θ. Fix any measurable subset of types W, any $\alpha_1, \alpha_2 \in \mathcal{A}$ with $\alpha_1(w) \le \alpha_2(w)$ for all $w \in W$, and any cutoff function $\kappa : [\underline{w}, \bar{w}] \to [\inf\Theta + \nu/2, \sup\Theta - \nu/2]$ such that*

$\alpha(w,s) = \alpha_1(w)$ if $s < \kappa(w)$ and $\alpha(w,s) = \alpha_2(w)$ if $s \geq \kappa(w)$ for all $w \in W$. Then,

$$\lim_{\nu\to 0}\int_W (\alpha_2(w)-\alpha_1(w))G_w(z|\kappa;\alpha,W)\frac{\phi(\kappa(w))f(w)}{\int_W \phi(\kappa(w))f(w)dw}dw = z - \bar{\alpha}(\alpha_1,W) \quad (24)$$

for all $z \in [\bar{\alpha}(\alpha_1,W), \bar{\alpha}(\alpha_2,W)]$, and the convergence as $\nu \to 0$ is uniform.

Proof of Lemma 5. The proof adapts the steps of the proof of Lemma 3 to the case of a non-uniform prior. Specifically, we first show that the joint density of extended types (s,w) conditional on $\tilde{s}=0$ and $w \in W$ uniformly converges to

$$\lim_{\nu\to 0} Pr(\kappa(w),w|\tilde{s}=0,W) = \frac{\phi(k(w))\hat{f}(w)}{\int_W \phi(\kappa(w))\hat{f}(w)dw}, \quad (25)$$

Given this, we show that $y(\kappa,W,\theta)$ is uniformly distributed in $[0,1]$ conditional on $\tilde{s}=0$ so that condition (23) holds in the limit. Accordingly, combining (25) and (23) we obtain (24).

To prove (25) note that the joint density of (s,w,θ) is now given by

$$Pr(s,w,\theta|W) = Pr(s|w,\theta)Pr(w|W)Pr(\theta) = \left(h_w\left(\frac{s-\theta}{\nu}\right)\frac{1}{\nu}\right)\left(\frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}\right)\phi(\theta).$$

We obtain the marginal density of (s,w) by integrating the above expression, which leads to, after applying the change of variable $\theta' = \frac{s-\theta}{\nu}$,

$$\begin{aligned} Pr(s,w|W) &= \int_{s-\nu/2}^{s+\nu/2} h_w\left(\frac{s-\theta}{\nu}\right)\frac{1}{\nu}\frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}\phi(\theta)d\theta \\ &= \int_{-1/2}^{1/2} h_w(\theta')\frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}\phi(s-\nu\theta')d\theta' \to \frac{\hat{f}(w)}{\int_W \hat{f}(w)dw}\phi(s) \text{ as } \nu \to 0. \end{aligned}$$

The limit is continuous, so pointwise convergence of distribution functions implies uniform convergence. We obtain condition (25) by taking the limit as $\nu \to 0$,

$$Pr(s=\tilde{s}+\kappa(w)|W) = \int_W Pr(\tilde{s}+\kappa(w),w|W)dw \to \frac{\int_W \phi(\tilde{s}+\kappa(w))\hat{f}(w)dw}{\int_W \hat{f}(w)dw}.$$

To show that (23) holds in the limit notice that the distribution of the virtual noise converges to the mixture distribution $\left\{H_w\left(\tilde{\eta}+\frac{\kappa(w)}{\nu}\right), \frac{\phi(\kappa(w))\hat{f}(w)}{\int_W \phi(\kappa(w))\hat{f}(w)dw}\right\}_W$, which does not depend on θ so $\tilde{\eta}$ is i.i.d. across agents and independent of θ. Hence, the argument in the last part of the proof of Lemma 3 applies. □

Appendix B Omitted Proofs

B.1 Proofs of Results in Section 3

Proof of Proposition 1. The "if" part of the proof is based on the following steps. First, we show that condition (3) implies that payoffs satisfy *externality symmetry*: the infinitesimal change in the payoff differences of type w when type w' switches actions is the same as the change in payoff differences of type w' when type w switches actions. Second, we show that externality symmetry implies quasilinearity.

For the "only if" part, we construct a potential function for quasilinear payoffs that satisfies condition (3).

"If" part: Assume that there exists functional V satisfying condition (3) and focus on how the infinitesimal change in V due to a switch of type w from a to a' changes when type w' switches from a'' to a'''. Let strategy profiles $\alpha, \alpha', \alpha''$ and α''' satisfy $\alpha(w) = \alpha''(w) = a$, $\alpha'(w) = \alpha'''(w) = a'$, $\alpha(w') = \alpha'(w') = a''$, $\alpha''(w') = \alpha'''(w') = a'''$ and $\alpha(w'') = \alpha'(w'') = \alpha''(w'') = \alpha'''(w'')$ for all $w'' \notin \{w, w'\}$.

Abusing notation, let $\bar{\alpha}_\epsilon(\alpha)$ denote the aggregate action under mixture distribution $(1-\epsilon)F + \epsilon\delta(w')$. Note that $\bar{\alpha}_\epsilon(\alpha) = \bar{\alpha}_\epsilon(\alpha')$ and $\bar{\alpha}_\epsilon(\alpha'') = \bar{\alpha}_\epsilon(\alpha''') = \bar{\alpha}_\epsilon(\alpha) + \epsilon(a''' - a'')$, where

$$\bar{\alpha}_\epsilon(\alpha) = (1-\epsilon)\int_{\underline{w}}^{\overline{w}} \alpha(w)dF(w) + \epsilon a'' = (1-\epsilon)\bar{\alpha}(\alpha) + \epsilon a''.$$

Given the mixture distribution $F_{ww'}^{\varepsilon\epsilon} = (1-\varepsilon-\epsilon)F + \varepsilon\delta(w) + \epsilon\delta(w')$ the definition of potential (3) implies that

$$\begin{aligned}
\lim_{\varepsilon\to 0} & \frac{(V(\alpha''', F_{ww'}^{\varepsilon\epsilon}, \theta) - V(\alpha'', F_{ww'}^{\varepsilon\epsilon}, \theta)) - (V(\alpha', F_{ww'}^{\varepsilon\epsilon}, \theta) - V(\alpha, F_{ww'}^{\varepsilon\epsilon}, \theta))}{\varepsilon} \\
&= \Delta U(a', a, \bar{\alpha}_\epsilon(\alpha''), \theta, w) - \Delta U(a', a, \bar{\alpha}_\epsilon(\alpha), \theta, w) \\
&= \Delta U(a', a, \bar{\alpha}_\epsilon(\alpha) + \epsilon(a''' - a''), \theta, w) - \Delta U(a', a, \bar{\alpha}_\epsilon(\alpha), \theta, w)
\end{aligned}$$

We can further divide this difference by ϵ and obtain

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{1}{\epsilon} \lim_{\varepsilon\to 0} & \frac{(V(\alpha''', F_{ww'}^{\varepsilon\epsilon}, \theta) - V(\alpha'', F_{ww'}^{\varepsilon\epsilon}, \theta)) - (V(\alpha', F_{ww'}^{\varepsilon\epsilon}, \theta) - V(\alpha, F_{ww'}^{\varepsilon\epsilon}, \theta))}{\varepsilon} \\
&= (a''' - a'')\frac{\partial \Delta U(a', a, \bar{\alpha}(\alpha), \theta, w)}{\partial \bar{a}}.
\end{aligned}$$

Note that the RHS is well defined since U, and hence ΔU, is differentiable with respect to $\bar{a}$. A similar argument shows that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\lim_{\epsilon\to 0}\frac{(V(\alpha''',F^{\varepsilon\epsilon}_{ww'},\theta)-V(\alpha',F^{\varepsilon\epsilon}_{ww'},\theta))-(V(\alpha'',F^{\varepsilon\epsilon}_{ww'},\theta)-V(\alpha,F^{\varepsilon\epsilon}_{ww'},\theta))}{\epsilon}$$
$$=(a'-a)\frac{\partial\Delta U(a''',a'',\bar{\alpha}(\alpha),\theta,w')}{\partial\bar{a}}.$$

The definition of potential states that both limits must coincide, yielding the following *externality symmetry* condition: for all a, a', a'' and a''', all α and all w, w',

$$(a'''-a'')\frac{\partial\Delta U(a',a,\bar{\alpha}(\alpha),\theta,w)}{\partial\bar{a}}=(a'-a)\frac{\partial\Delta U(a''',a'',\bar{\alpha}(\alpha),\theta,w')}{\partial\bar{a}}. \tag{26}$$

We next prove that (26) requires payoffs to be additively separable in $\bar{a}$ and w. That is, they must take on the form $U(a,\bar{a},\theta,w)=\mathsf{u}(a,\bar{a},\theta)+v(a,\theta,w)+u_0(\bar{a},\theta,w)$. If we set $a''=a$ and $a'''=a'$, (26) implies that

$$\frac{\partial\Delta U(a',a,\bar{\alpha}(\alpha),\theta,w)}{\partial\bar{a}}=\frac{\partial\Delta U(a',a,\bar{\alpha}(\alpha),\theta,w')}{\partial\bar{a}},$$

for all α and all w, w'. Hence, payoffs must be separable since the partial derivative of payoff differences w.r.t. the aggregate action is independent of types.

Consider next the linearity of $\mathsf{u}(a,\bar{a},\theta)$ w.r.t. a. First, if there are just two actions $a<a'$ in A then we can always write separable payoffs in a linear form $au(\bar{a},\theta)$ by defining $u(\bar{a},\theta)=\frac{1}{a'-a}\left(\mathsf{u}(a',\bar{a},\theta)-\mathsf{u}(a,\bar{a},\theta)\right)$ and adding to u_0 the term $\frac{a'}{a'-a}\mathsf{u}(a,\bar{a},\theta)+\frac{a}{a'-a}\mathsf{u}(a',\bar{a},\theta)$. In general, when there are more than two actions, setting $w=w'$ in expression (26) implies that

$$\frac{\partial}{\partial\bar{a}}\frac{\Delta U(a,a',\bar{a},\theta,w)}{(a-a')}=\frac{\partial}{\partial\bar{a}}\frac{\Delta U(a''',a'',\bar{a},\theta,w)}{(a'''-a'')},$$

i.e., $\frac{\partial}{\partial\bar{a}}\frac{\Delta U(a,a',\bar{a},\theta,w)}{\Delta a}$ is independent of a and a'. Accordingly, we can write u as a linear function of a.

Finally, since the existence of potential implies the above separability and linearity restrictions, for Γ_θ to be a weighted potential game, there must exist a function $\psi(\theta,w)$ such that $\psi(\theta,w)U(a,\bar{a},\theta,w)=\mathsf{u}(a,\bar{a},\theta)+v(a,\theta,w)+u_0(\bar{a},\theta,w)$. But this implies that $U(a,\bar{a},\theta,w)$ satisfies (1), where $c(\theta,w)=1/\psi(\theta,w)>1/\zeta=\xi>0$. That is, payoffs must be quasilinear.

"Only If" part: Given quasilinear payoffs consider the functional defined by (5). The change in V under mixture distribution F^{ε}_{w} when type w switches from action $\alpha(w)$

to a is given by

$$\int_{(1-\varepsilon)\bar{\alpha}(\alpha)+\varepsilon\alpha(w)}^{(1-\varepsilon)\bar{\alpha}(\alpha)+\varepsilon a} u(z,\theta)dz + \varepsilon\left(v(a,\theta,w) - v(\alpha(w),\theta,w)\right).$$

Dividing by ε and taking the limit, we obtain (3):

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_{(1-\varepsilon)\bar{\alpha}(\alpha)+\varepsilon\alpha(w)}^{(1-\varepsilon)\bar{\alpha}(\alpha)+\varepsilon a} u(z,\theta)dz + \varepsilon\left(v(a,\theta,w) - v(\alpha(w),\theta,w)\right)\right). \\
&\quad = \frac{\partial}{\partial\varepsilon}\int_{(1-\varepsilon)\bar{\alpha}(\alpha)+\varepsilon\alpha(w)}^{(1-\varepsilon)\bar{\alpha}(\alpha)+\varepsilon a} u(z,\theta)dz + v(a',\theta,w) - v(\alpha(w),\theta,w) \\
&\quad = (a-\alpha(w))u(\bar{\alpha}(\alpha),\theta) + v(a',\theta,w) - v(\alpha(w),\theta,w) = \Delta U(a,\alpha(w),\bar{\alpha}(\alpha),\theta,w).
\end{aligned}$$

That is, the change in V in the limit coincides with the change in payoffs for type w, up to scaling by a function $c(\theta,w)$ so V is a (weighted) potential of Γ_θ. In addition, given that such a change in V is differentiable with respect to ε, it is straightforward to check that the double limit (4) exists and is independent of the order of taking limits. □

B.2 Proofs of Results in Subsection 4.1

We use the following result to prove the existence of a potential maximizing strategy profile. Let $\{W_j^n\}_{j=1}^n$ be a partition of $[\underline{w},\overline{w}]$ into n intervals W_j^n of the same length and define the set of strategy profiles that assign the same action to types in each W_j as

$$\mathcal{A}^n = \{\alpha : \alpha(w) = a_j \in A \text{ for all } w \in W_j^n,\ j = 1,\cdots,n\}.$$

Lemma 6. *For any $\alpha \in \mathcal{A}$ and any $\varepsilon > 0$ there exist a sequence $\{\alpha^n\}$ with $\alpha^n \in \mathcal{A}^n$ that converges pointwise to α as $n \to \infty$ in a subset of types with measure at least $1-\varepsilon$.*

Proof. A continuous function on $[\underline{w},\overline{w}]$ can be approximated pointwise by a sequence of step functions on A^n. By Lusin's theorem, for any $\varepsilon > 0$ a measurable function on $[\underline{w},\overline{w}]$ is continuous in a compact subset of $[\underline{w},\overline{w}]$ with Lebesgue measure at least $\overline{w} - \underline{w} - \varepsilon$. Accordingly, since F is a continuous distribution, for any $\alpha \in \mathcal{A}$ and any $\varepsilon > 0$ we can find a sequence of step functions $\{\alpha^n\}$ that converges pointwise to α in a subset of types with measure at least $1-\varepsilon$. □

Proof of Proposition 2. To prove the first part we show that the problem of maximizing potential coincides with the problem of maximizing ex-ante payoffs under marginal beliefs given by (6). Maximizing potential implies finding a strategy profile that solves

$$\max_{\alpha\in\mathcal{A}} \int_0^{\bar{\alpha}(\alpha)} u(z,\theta)dz + \int_{\underline{w}}^{\overline{w}} v(\alpha(w),\theta,w)dF(w). \tag{27}$$

Consider the change of variable $z = \bar{\alpha}(\alpha,[w,\bar{w}])(1-F(w))$ to the first integral. Differentiating this expression w.r.t. w we obtain $dz = -\alpha(w)f(w)dw$. In addition, $0 = \bar{\alpha}(\alpha,[\overline{w},\overline{w}])$ and $\bar{\alpha}(\alpha) = \bar{\alpha}(\alpha,[\underline{w},\overline{w}])$, which leads to objective function

$$\int_{\underline{w}}^{\overline{w}} \alpha(w)u(\bar{\alpha}(\alpha,[w,\bar{w}])(1-F(w)),\theta)f(w)dw + \int_{\underline{w}}^{\overline{w}} v(\alpha(w),\theta,w)dF(w).$$

To prove that α^* must coincide a.e. with a NE assume, by way of contradiction, that α^* maximizes potential but that there is a closed set of types of positive measure that is not best responding to α^*. Accordingly, there must be an interval W such that almost all types $w\in W$ are not best responding, i.e., $\Delta U(\alpha^*(w),a,\bar{\alpha}(\alpha^*),\theta,w)<0$ for some $a\in A$. We can then find a measurable strategy profile α, with $\alpha(w')=\alpha^*(w')$ for all $w'\notin W$, such that $\Delta U(\alpha^*(w),\alpha(w),\bar{\alpha}(\alpha^*),\theta,w)<0$ for almost all $w\in W$. This is because we can create a convergent sequence of step functions by partitioning W into n equal-sized intervals and assigning to all types in each subinterval the action that maximizes U given $\bar{\alpha}(\alpha^*)$ for the middle type in the subinterval. For n large enough, such a strategy yields a higher payoff to almost all $w\in W$ since U is Lipschitz continuous.[18]

The difference in potential between α^* and α can be written as

$$V(\alpha_P,\theta)-V(\alpha,\theta) = \int_0^{\bar{\alpha}(\alpha^*)-\bar{\alpha}(\alpha)} u(\bar{\alpha}(\alpha)+z,\theta)dz + \int_W \left(v(\alpha^*(w),\theta,w)-v(\alpha(w),\theta,w)\right)dF(w).$$

Let $z(w)=\int_{w'\in W}\mathbf{1}_{\{w'\geq w\}}(\alpha^*(w')-\alpha(w'))dF(w')$. Differentiating w.r.t. w we obtain $dz=-(\alpha^*(w)-\alpha(w))dF(w)$. Also, $z(\min W)=\bar{\alpha}(\alpha^*)-\bar{\alpha}(\alpha)$ and $z(\max W)=0$.

[18] The measurability of α comes from the fact that it is the linear combination of a step function and the restriction of a measurable function to the measurable set $[\underline{w},\overline{w}]\setminus W$.

Hence, applying a change of variable we get that

$$
\begin{aligned}
&V(\alpha^*,\theta) - V(\alpha,\theta)\\
&\qquad = \int_W (\alpha^*(w) - \alpha(w)) u\Big(\bar{\alpha}(\alpha) + \int_W \mathbf{1}_{\{w' \geq w\}}(\alpha^*(w') - \alpha(w'))dF(w'), \theta\Big) dF(w)\\
&\qquad\quad + \int_W \left(v(\alpha^*(w),\theta,w) - v(\alpha(w),\theta,w)\right) dF(w)\\
&\qquad = \int_W \Delta U\Big(\alpha^*(w), \alpha(w), \bar{\alpha}(\alpha) + \int_W \mathbf{1}_{\{w' \geq w\}}(\alpha^*(w') - \alpha(w'))dF(w'), \theta, w\Big) dF(w).
\end{aligned}
$$

Note that, as the mass of W vanishes, $\int_W \mathbf{1}_{\{w' \geq w\}}(\alpha^*(w') - \alpha(w'))dF(w')$ goes to zero and $\bar{\alpha}(\alpha)$ uniformly converges to $\bar{\alpha}(\alpha^*)$, implying that the integrand in the above expression converges to $\Delta U(\alpha^*(w), \alpha(w), \bar{\alpha}(\alpha^*), \theta, w)$ for all $w \in W$. Since ΔU is Lipschitz continuous and $\Delta U(\alpha^*(w), \alpha(w), \bar{\alpha}(\alpha^*), \theta, w) < 0$ for almost all $w \in W$, we can always find an interval W with small enough probability mass so that

$$
\Delta U\Big(\alpha^*(w), \alpha(w), \bar{\alpha}(\alpha) + \int_W \mathbf{1}_{\{w' \geq w\}}(\alpha^*(w') - \alpha(w'))dF(w'), \theta, w\Big) < 0
$$

for almost all $w \in W$. But this implies that $V(\alpha^*,\theta) - V(\alpha,\theta) < 0$, a contradiction. We conclude that α^* implies best responding for almost all types.

Next we argue that we can switch the strategies of only those types that are not best responding under α^* to obtain a measurable strategy profile that is both a NE and a potential maximizer. First note that changing the strategy of types that are not maximizing payoffs is not going to affect aggregate action $\bar{\alpha}(\alpha^*)$ and hence the payoffs of other players, so the latter would still be best responding under the modified profile. In addition, if type w is not maximizing her payoff by playing $\alpha^*(w)$ we can replace her strategy by any strategy satisfying

$$
\alpha'(w) \in \arg\max_{a \in A} U(a, \bar{\alpha}(\alpha^*), \theta, w).
$$

A is compact and U is continuous so this maximization problem has a solution. Since the Lebesgue measure on $[\underline{w}, \overline{w}]$ is complete, the modified profile is also measurable since it coincides with a measurable function a.e. Accordingly, every type would be best responding under the modified strategy profile, i.e., it represents a NE of Γ_θ. Moreover, it maximizes potential given that it coincides a.e. with α^*. This also implies that the existence of a potential maximizing profile α^* implies the existence of a NE.

Finally, we prove that set of solutions to the problem of maximizing potential given by (27) is non-empty. To do so we first show that, if we restrict the maximization problem to strategies given by step functions in $\mathcal{A}^n$, the set of potential maximizers is non-empty. Subsequently, we use the fact that measurable strategies can be approximated by step functions (Lemma 6) to argue the existence of a measurable strategy profile that maximizes potential.

Let R^n be endowed with the product topology. Note that $\mathcal{A}^n$ is a subset of $\mathcal{A}$ and coincides with the product space $A^n \subset \mathbb{R}^n$, which is compact since A is compact. In addition, the continuity and boundedness of u, v and F implies that V is well-defined, bounded and continuous in $\mathcal{A}^n$. Hence, by Weiertrass extreme value theorem, $\max_{\alpha \in \mathcal{A}^n} V(\alpha, \theta)$ has a solution for any $n < \infty$. Let $V^n := \max_{\alpha \in \mathcal{A}^n} V(\alpha, \theta)$.

The sequence $\{V^{2^n}\}$ converges as $n \to \infty$. This is because the subsets in $\{W_j^{2^n}\}_{j=1}^{2^n}$ are unions of subsets in $\{W_j^{2^{n+1}}\}_{j=1}^{2^{n+1}}$, implying that $\mathcal{A}^{2^n} \subset \mathcal{A}^{2^{n+1}}$ and thus $V^{2^{n+1}} \geq V^{2^n}$ for all n. Hence, $\{V^{2^n}\}$ is a monotone bounded sequence so it must converge. Let $L = \lim V^{2^n}$. Since $\mathcal{A}^{2^n} \subset \mathcal{A}$, there exist $\alpha \in \mathcal{A}$ such that $V(\alpha, \theta) \geq V^{2^n}$ for all n, implying that $V(\alpha, \theta) \geq L$ since $\{V^{2^n}\}$ is a convergent sequence.

To finish the proof we argue that there does not exist $\alpha' \in \mathcal{A}$ such that $V(\alpha', \theta) > L$. Assume that there is such an α'. Then by Lemma 6 we can find a sequence $\{\alpha'^{2^n}\}$ with $\alpha'^{2^n} \in \mathcal{A}^{2^n}$ that converges pointwise to α' except in a zero measure set. This implies that $\lim \bar{\alpha}(\alpha'^{2^n}) = \bar{\alpha}(\alpha')$ and, by the Lipschitz continuity of u, v and the continuity of F,

$$\lim_{n\to\infty} \int_0^{\bar{\alpha}(\alpha'^{2^n})} u(z, \theta) dz + \lim_{n\to\infty} \int_{\underline{w}}^{\overline{w}} v(\alpha'^{2^n}(w), \theta, w) dF(w)$$
$$= \int_0^{\bar{\alpha}(\alpha')} u(z, \theta) dz + \int_{\underline{w}}^{\overline{w}} v(\alpha(w)', \theta, w) dF(w).$$

But then $L = \lim V^{2^n} \geq \lim V(\alpha'^{2^n}, \theta) = V(\alpha', \theta) > L$, a contradiction. □

Proof of Proposition 3. Existence of at least one NE that maximizes potential is guaranteed by Proposition 2. In addition, payoffs exhibiting strictly increasing differences in a and w implies that best responses are non-decreasing in w for any fixed $\bar{a}$. Hence, NE strategy profiles must be (weakly) increasing in w. Since any potential maximizer coincides a.e. with a NE by Proposition 2, a potential maximizing profile must also be increasing a.e. in $[\underline{w}, \overline{w}]$.

To show that there is an essentially unique potential maximizer it suffices to show that there is an essentially unique potential maximizing NE for each θ, except perhaps in a countable subset of Θ. We do so by showing that, if at some θ there are two or more potential maximizing NE that differ in a positive measure set of types, then there exist $\theta' < \theta$ and $\theta'' > \theta$ such the potential maximizing NE is essentially unique in $(\theta', \theta) \cup (\theta, \theta'')$. Since Θ can only be partitioned in a countable number of non-degenerate intervals then the set of θ at which the potential maximizing NE is not essentially unique must be countable, i.e., it must have Lebesgue measure zero.

First, we argue that there cannot be two NE at θ that exhibit the same aggregate action $\bar{a}^*$ but differ in a positive measure set of types. To see why, assume that α_1 and α_2 are two NE with the same aggregate action $\bar{a}^*$. If they were to differ in a positive measure of types W it must be that $\alpha_1(w)$ and $\alpha_2(w)$ yield the same level of utility, i.e., $\alpha_1(w)u(\bar{a}^*, \theta) + v(\alpha_1(w), \theta, w) = \alpha_2(w)u(\bar{a}^*, \theta) + v(\alpha_2(w), \theta, w)$ for all $w \in W$. This implies that

$$\frac{v(\alpha_1(w), \theta, w) - v(\alpha_2(w), \theta, w)}{\alpha_1(w) - \alpha_2(w)} = -u(\bar{a}^*, \theta)$$

for all $w \in W$. But notice that the LHS of this equality is strictly increasing in w since payoffs exhibit strict increasing differences by condition (i) in Assumption 1 while the RHS is constant in w. Accordingly, there cannot exist two types for which the equality holds and hence all types, except possibly one, have a unique best response to aggregate action $\bar{a}^*$.

Second, strict increasing differences w.r.t. a and $\bar{a}$ imply that NE are ordered. Specifically, if there exist two NE profiles α_1 and α_2 at θ, with α_2 having a higher aggregate action than α_1, then $\alpha_2(w) \geq \alpha_1(w)$ for all w, with strict inequality for a positive mass of types. This is because condition (i) in Assumption 1 ensures that players' best responses are increasing in the aggregate action.

Third, we show that, for any two increasing strategy profiles satisfying $\alpha_2(w) \geq \alpha_1(w)$ for all w with strict inequality for a positive measure of types, the difference in potential is increasing in θ. By Proposition 2 the difference in potential is given by

$$\int_{\underline{w}}^{\overline{w}} \Big(U\big(\alpha_2(w), \bar{\alpha}(\alpha_2, [w, \overline{w}])(1 - F(w)), \theta, w\big) \\ - U\big(\alpha_1(w), \bar{\alpha}(\alpha_1, [w, \overline{w}])(1 - F(w)), \theta, w\big)\Big) dF(w).$$

Note that $\bar{\alpha}(\alpha_2, [w, \overline{w}]) \geq \bar{\alpha}(\alpha_1, [w, \overline{w}])$ for all w (except possibly for the highest type), with strict inequality for a positive mass of types. By condition (ii) in Assumption 1 the integrand is increasing in θ for all w s.t. $\alpha_2(w) \geq \alpha_1(w)$ and $\bar{\alpha}(\alpha_2, [w, \overline{w}]) \geq \bar{\alpha}(\alpha_1, [w, \overline{w}])$, and strictly so if one of the inequalities is strict. Since these inequalities are satisfied for all types and strictly so for a positive measure of them, the difference in potential between α_2 and α_1 is strictly increasing in θ.

Now assume that there exist multiple potential-maximizing NE at θ and let α^* be the largest of them, i.e., the one exhibiting the highest individual actions for all types. Next, consider an infinitesimal increase in θ. Such an increase leads to α^* yielding a higher potential than any smaller strategy profile, since their differences in potential increase with θ. This breaks the tie among all potential maximizing profiles in favor of α^*. Next, for any small $\varepsilon > 0$ consider any profile $\alpha > \alpha^*$ such that $\alpha(w) - \alpha^*(w) > \varepsilon$ for all w in a set of types W of measure at least ε.

Since α^* is the largest potential maximizer at θ, the Lipschitz continuity of u, v and the continuity of F implies that α^* still yields a higher potential than α after an infinitesimal increase in θ. Accordingly, such an increase leads to an essentially unique potential maximizer.

A symmetric argument applies to the case of an infinitesimal drop in θ. Accordingly, there is an essentially unique maximizer in an open neighborhood $(\theta', \theta) \cup (\theta, \theta'')$.

The existence of the mapping α_P directly follows from the monotonicity of NE strategies w.r.t. w and the essential uniqueness of potential-maximizing NE. □

B.3 Proofs of Results in Subsection 4.2

The proofs of Propositions 4 and 5 first focus on the uniform prior case and then resort to the following lemma about the uniform convergence of individual beliefs as noise vanishes to extend the results to any well-defined prior.

Lemma 7. *Let $J_{w'|w}(s'|s;\nu,\phi)$ denote the cdf of signals of type w' agents conditional on an agent of type w receiving signal s, for given noise level ν and common prior ϕ. Given any $s \in [\inf\Theta + \nu, \sup\Theta - \nu]$ and any sequence s^ν such that $\frac{s-s^\nu}{\nu} = c$ for some constant $|c| < 1$, as $\nu \to 0$, $|J_{w'|w}(s^\nu|s;\nu,\phi) - J_{w'|w}(s^\nu|s;\nu,U[\inf\Theta,\sup\Theta])|$ converges uniformly to zero.*

Proof of Lemma 7. Let s_w denote the random variable representing the signals received by agents of type w. The beliefs about $s_{w'}$ of an agent of type w conditional on receiving s can be expressed as

$$J_{w'|w}(s'|s;\nu,\phi) = \frac{Pr(s'_w < s', s_w = s)}{Pr(s_w = s)} = \frac{\int_{s-\nu/2}^{s+\nu/2} H_{w'}\left(\frac{s'-\theta}{\nu}\right) h_w\left(\frac{s-\theta}{\nu}\right) \frac{1}{\nu}\phi(\theta)d\theta}{\int_{s-\nu/2}^{s+\nu/2} h_w\left(\frac{s-\theta}{\nu}\right) \frac{1}{\nu}\phi(\theta)d\theta}$$

Using the change of variable $\theta' = \frac{s-\theta}{\nu}$, for any $s' = s - c\nu$ we have that

$$J_{w'|w}(s'|s;\nu,\phi) = \frac{\int_{-1/2}^{1/2} H_{w'}\left(\theta' - c\right) h_w\left(\theta'\right) \phi(s-\nu\theta')d\theta'}{\int_{-1/2}^{1/2} h_w\left(\theta'\right) \phi(s-\nu\theta')d\theta'}$$

$$\to \int_{-1/2}^{1/2} H_{w'}\left(\theta' - c\right) h_w\left(\theta'\right) d\theta' = J_{w'|w}(s'|s;\nu,U[\inf\Theta,\sup\Theta]),$$

as $\nu \to 0$. Since $J_{w'|w}$ is a cdf and the limit is continuous, pointwise convergence implies uniform convergence. $\square$

Proof of Proposition 4. The proof logic is as follows. First, we argue for the uniform prior case that, given any $\nu > 0$, the set of equilibrium strategy profiles has a largest and a smallest element, each involving monotone strategies.

Second, we show that there is at most one equilibrium in monotone strategies, up to differences in behavior at discontinuities (signal cutoffs). Since increasing functions have at most a countable number of discontinuities, the smallest and largest equilibria are essentially the same, that is, they coincide for each w except possibly in a zero measure set of signals. These arguments extend to the non-uniform prior case by the uniform convergence of beliefs (Lemma 7).

Consider the following 'ex-ante' version of the game in which the (infinitesimal) mass of players with type w and signal $s = \theta + \nu\eta$ is determined by the joint distribution of (w, θ, η), which has density $h_w(\eta)f(w)\frac{1}{\overline{\theta}-\underline{\theta}}$. A strategy profile in this game is a measurable function $\alpha : [\underline{w}, \overline{w}] \times [\underline{\theta} - \nu/2, \overline{\theta} + \nu/2] \to A$. Let $\bar{\alpha}^\nu(\alpha; \theta)$ denote the average action given α and θ when the noise scale is ν. It is given by

$$\bar{\alpha}^\nu(\alpha; \theta) = \int_{\underline{w}}^{\overline{w}} \int_{-1/2}^{1/2} \alpha(w, \theta + \nu\eta)h_w(\eta)d\eta dF(w). \tag{28}$$

The payoff of a player of type w and signal s that takes action a is given by $E[U(a, \bar{\alpha}^\nu(\alpha; \theta), \theta, w)|s]$, where the expectation is taken over θ conditional on s. This game is identical to the Bayesian game in which agents have a common uniform prior about θ and the exact LLN applies within each type. Hence, the set of NE in the game corresponds to the set of Bayesian NE of the Bayesian game.

This game satisfies the definition of supermodular game in Milgrom and Roberts (1990). Specifically, since we restrict attention to measurable strategy profiles the strategy space is a complete lattice. In addition, payoff functions are order-continuous in a and in α, and they exhibit increasing differences by Assumption 1.[19] Accordingly, Theorem 5 in Milgrom and Roberts (1990) implies that the game has a

[19] $E[U(a, \bar{\alpha}^\nu(\alpha; \theta), \theta, w)|s]$ is order-continuous in α if

$$\lim_{\alpha \in C, \alpha \downarrow \inf(C)} E[U(a, \bar{\alpha}^\nu(\alpha; \theta), \theta, w)|s] = E[U(a, \bar{\alpha}^\nu(\inf(C); \theta), \theta, w)|s]$$

and

$$\lim_{\alpha \in C, \alpha \uparrow \sup(C)} E[U(a, \bar{\alpha}^\nu(\alpha; \theta), \theta, w)|s] = E[U(a, \bar{\alpha}^\nu(\sup(C); \theta), \theta, w)|s]$$

for any chain C in the set of measurable strategy profiles. This is true because U is Lipschitz continuous in the average action and the distributions of θ and (s, w) conditional on s are continuous, which imply that $\lim_{\alpha \in C, \alpha \downarrow \inf(C)} \bar{\alpha}^\nu(\alpha; \theta) = \bar{\alpha}^\nu(\inf(C); \theta)$ and $\lim_{\alpha \in C, \alpha \uparrow \sup(C)} \bar{\alpha}^\nu(\alpha; \theta) = \bar{\alpha}^\nu(\sup(C); \theta)$ for all $\theta \in \Theta$.

smallest equilibrium α^l and a largest equilibrium and α^m such that any equilibrium profile α satisfies $\alpha^l(w,s) \leq \alpha(w,s) \leq \alpha^m(w,s)$.

In addition, fixing the actions of all agents, an agent's difference in expected payoffs conditional on s from choosing a versus $a' < a$ is increasing in s since the aggregate action is kept fixed while θ is higher (in expectation) at higher signal profiles. That is, expected payoffs exhibit increasing differences w.r.t. a and the profile of signals for every w. Hence, Theorem 6 in Milgrom and Roberts (1990) applies: the smallest and largest equilibria are nondecreasing w.r.t. the profile of signals. Since an agent's strategy can only depend on her own signal, this implies that α^l and α^m are monotone functions of s. A similar argument applies to monotonicity w.r.t. w.

To show that there is at most one equilibrium in monotone strategies, we first establish the following translation result. Given $\delta > 0$, let α_δ represent a "rightward shift" of strategy profile α defined by

$$\alpha_\delta(w,s) = \begin{cases} \alpha(w, \inf\Theta - \nu/2) & s < \inf\Theta - \nu/2 + \delta \\ \alpha(w, s-\delta) & s \geq \inf\Theta - \nu/2 + \delta. \end{cases}$$

The next lemma shows that if we simultaneously switch agents' strategies from α to α_δ and their signals from s to $s+\delta$ then an agent's conditional expectation of payoff differences between a higher action a and a lower action $a' < a$ strictly increases. [We omit the dependence of α on ν to ease notation.]

Lemma 8. *There exists $\bar{\nu} > 0$ such that, for any $\alpha \in \mathcal{A}$, if $\nu < \bar{\nu}$ then*

$$E[\Delta U(a,a',\bar{\alpha}^\nu(\alpha_\delta;\theta),\theta,w)|s+\delta] - E[\Delta U(a,a',\bar{\alpha}^\nu(\alpha;\theta),\theta,w)|s] \geq K(a-a')\delta \quad (29)$$

for all actions a and $a' < a$, all $w \in [\underline{w}, \bar{w}]$, all $s \in [\underline{\theta} - \nu/2, \overline{\theta} + \nu/2]$ and all $\delta \in (0, \sup\Theta - s - \nu/2]$.

Proof. For all $\nu < \bar{\nu} := \min\{\underline{\theta} - \inf\Theta, \sup\Theta - \bar{\theta}\}$, the support of the distributions of θ and other player signals conditional on $s \in [\underline{\theta} - \nu/2, \overline{\theta} + \nu/2]$ are respectively $[s-\nu/2, s+\nu/2]$ and $[s-\nu, s+\nu]$. In such a case, the conditional density of θ is $h_w\left(\frac{s-\theta}{\nu}\right)/\nu$. Also notice that, conditional on θ, the signals of other agents are independent of s, with densities $h_{w'}\left(\frac{s'-\theta}{\nu}\right)/\nu$. Accordingly, aggregate action (28) can be expressed as

$$\bar{\alpha}^\nu(\alpha;\theta) = \int_{\underline{w}}^{\overline{w}} \int_{\theta-\nu/2}^{\theta+\nu/2} \alpha(w,s') h_w\left(\frac{s'-\theta}{\nu}\right) \frac{1}{\nu} ds' dF(w'). \quad (30)$$

By Assumption 1, for any $s \in [\underline{\theta} - \nu/2, \overline{\theta} + \nu/2]$ and $\delta \in (0, \sup\Theta - s - \nu/2]$ we obtain the following inequality using the changes of variables $\theta' = \theta + \delta$ and $s'' = s' + \delta$:

$$E[\Delta U(a, a', \bar{\alpha}^\nu(\alpha; \theta), \theta, w)|s] + K(a - a')\delta =$$

$$\int_{s+\nu/2}^{s-\nu/2} \Delta U\left(a, a', \int_{\underline{w}}^{\overline{w}} \int_{\theta-\nu/2}^{\theta+\nu/2} \alpha(w', s') h_{w'}\left(\frac{s'-\theta}{\nu}\right) \frac{ds'}{\nu} dF(w'), \theta, w\right) h_w\left(\frac{s-\theta}{\nu}\right) \frac{d\theta}{\nu} + K(a-a')\delta$$

$$\leq \int_{s+\nu/2}^{s-\nu/2} \Delta U\left(a, a', \int_{\underline{w}}^{\overline{w}} \int_{\theta-\nu/2}^{\theta+\nu/2} \alpha(w', s') h_{w'}\left(\frac{s'-\theta}{\nu}\right) \frac{ds'}{\nu} dF(w'), \theta+\delta, w\right) h_w\left(\frac{s-\theta}{\nu}\right) \frac{d\theta}{\nu}$$

$$= \int_{s+\delta+\nu/2}^{s+\delta-\nu/2} \Delta U\left(a, a', \int_{\underline{w}}^{\overline{w}} \int_{\theta'-\delta-\nu/2}^{\theta'-\delta+\nu/2} \alpha(w', s') h_{w'}\left(\frac{s'+\delta-\theta'}{\nu}\right) \frac{ds'}{\nu} dF(w'), \theta', w\right) h_w\left(\frac{s+\delta-\theta'}{\nu}\right) \frac{d\theta'}{\nu}$$

$$= \int_{s+\delta+\nu/2}^{s+\delta-\nu/2} \Delta U\left(a, a', \int_{\underline{w}}^{\overline{w}} \int_{\theta'-\nu/2}^{\theta'+\nu/2} \alpha(w', s''-\delta) h_{w'}\left(\frac{s''-\theta'}{\nu}\right) \frac{ds''}{\nu} dF(w'), \theta', w\right) h_w\left(\frac{s+\delta-\theta'}{\nu}\right) \frac{d\theta'}{\nu}$$

$$= E[\Delta U(a, a', \bar{\alpha}^\nu(\alpha_\delta; \theta), \theta, w)|s+\delta].$$ [20] $\square$

Next, note that if α is an equilibrium then $\alpha(w, s) = 0$ for all $s \leq \underline{\theta} - \nu/2$ and $\alpha(w, s) = a_{max}$ if $s > \overline{\theta} + \nu/2$. This is because, for any action $a > 0$, Assumption 2 implies that $\Delta U(a, 0, \theta, w) < 0$ for all $\theta < \underline{\theta}$. Hence, since $\theta \leq s + \nu/2$, it must be that $E[\Delta U(a, 0, \bar{\alpha}^\nu(\alpha; \theta), \theta, w)|s] < 0$ for all $s < \underline{\theta} - \nu/2$. A symmetric argument applies to signals above $\bar{\theta} + \nu/2$.

We finish the proof by arguing that $\alpha^l(w, s) = \alpha^m(w, s)$ for almost all (w, s) using the above translation result. Assume first, by way of contradiction, that $\alpha^l(w, s) < \alpha^m(w, s)$ for some signal s and some type w and that there exists a signal shift $\delta > 0$ such that $\alpha^l(w, s + \delta) = \alpha^m(w, s)$ or, if $\alpha^m(w, \cdot)$ is discontinuous at s, $\alpha^l(w, s + \delta) \in [\liminf\ \alpha^m(w, s), \limsup\ \alpha^m(w, s)]$. Note that the monotonicity of $\alpha^l(w, \cdot)$ and $\alpha^m(w, \cdot)$ means that their $\liminf$ and $\limsup$ exist. Next, consider among all pairs (w, s) at which the two equilibria differ the largest signal shift $\hat{\delta}$ that

[20]The change of variable $\theta' = \theta + \delta$ works as long as the lower integration limit is well-defined, i.e., $s + \delta + \nu/2 \leq \sup \Theta$, which is guaranteed by $\delta \leq \sup \Theta - s - \nu/2$. The change of variable $s'' = s' + \theta$ works as long as $\theta' - \delta \geq \inf \Theta$, otherwise $\alpha(s'' - \delta, w')$ is not well-defined. Since $\theta' \geq s + \delta - \nu/2$ and $s \geq \underline{\theta} - \nu/2$ we have that $\theta' - \delta \geq \underline{\theta} - \nu$. Hence, $\theta' - \delta \geq \inf \Theta$ for any $\nu < \underline{\nu}$.

would be required to make $\alpha^l(w,s)$ 'equal' to $\alpha^m(w,s)$. That is $\hat{\delta}$ is given by

$$\hat{\delta} = \max \Big\{ \delta : \alpha^l(w, s+\delta) \in [\liminf \alpha^m(w,s), \limsup \alpha^m(w,s)] \\ \text{for some } (w,s) \text{ s.t. } \alpha^l(w,s+\delta) \neq \alpha^m(w,s+\delta) \Big\}.$$

It is straightforward to check that $\alpha^m(w, s-\hat{\delta}) \leq \alpha^l(w,s)$ for all s, w. Let $(\hat{w}, \hat{s})$ be one of the signal-type pairs associated with $\hat{\delta}$. Note that, by the above argument, we must have that $\hat{s} + \hat{\delta} \in [\underline{\theta} - \nu/2, \overline{\theta} + \nu/2]$, otherwise $\alpha^l(\hat{s}+\hat{\delta}, \hat{w}) = \alpha^m(\hat{s}+\hat{\delta}, \hat{w}) \in \{0, a_{max}\}$. Also note that if α is an equilibrium we must have that $E[\Delta U(a, \alpha(w,s), \bar{\alpha}^\nu(\alpha;\theta), \theta, w)|s] \leq 0$ for all $a \in A$. Hence, by Lemma 8, we arrive to the following contradiction:

$$\begin{aligned}
0 &\leq E[\Delta U(\alpha^m(\hat{w},\hat{s}), \alpha^l(\hat{w},\hat{s}), \bar{\alpha}^\nu(\alpha^m;\theta), \theta, \hat{w})|\hat{s}] \\
&< E[\Delta U(\alpha^m(\hat{w},\hat{s}), \alpha^l(\hat{w},\hat{s}), \bar{\alpha}^\nu(\alpha^m_{\hat{\delta}};\theta), \theta, \hat{w})|\hat{s}+\hat{\delta}] \\
&\leq E[\Delta U(\alpha^m(\hat{w},\hat{s}), \alpha^l(\hat{w},\hat{s}), \bar{\alpha}^\nu(\alpha^l;\theta), \theta, \hat{w})|\hat{s}+\hat{\delta}] \\
&\leq E[\Delta U(\alpha^m(\hat{w},\hat{s}), \alpha^l(\hat{w},\hat{s}+\hat{\delta}), \bar{\alpha}^\nu(\alpha^l;\theta), \theta, \hat{w})|\hat{s}+\hat{\delta}] \leq 0.
\end{aligned}$$

The third inequality comes from Assumption 1 and the fact that $\alpha^m(\hat{w},\hat{s}) > \alpha^l(\hat{w},\hat{s})$ and $\bar{\alpha}(\alpha^l) \geq \bar{\alpha}(\alpha^m_{\hat{\delta}})$ since $\alpha^m(w, s-\hat{\delta}) \leq \alpha^l(w,s)$ for all s, w. The last two inequalities follow from $\alpha^l(\hat{w}, \hat{s}+\delta)$ being a best response of type $\hat{w}$ to α^l under $\hat{s}+\hat{\delta}$.

Hence, the only possibility left for $\alpha^l(w,s) < \alpha^m(w,s)$ is that there is no signal shift $\delta > 0$ that satisfies $\alpha^l(w, s+\delta) \in [\liminf\ \alpha^m(w,s), \limsup \alpha^m(w,s)]$. Given that both $\alpha^l(w,\cdot)$ and $\alpha^m(w,\cdot)$ are increasing this can only happen in the set of s at which they are discontinuous, which has zero measure. □

Proof of Proposition 5. We first prove under a uniform prior that equilibrium strategy profiles converge pointwise a.e. to some strategy profile α_G. Specifically, we show that if α^ν is a sequence of equilibrium strategy profiles in the global game indexed by $\nu \to 0$, given any $\varepsilon > 0$ then there exists $\bar{\nu} > 0$ such that, for all ν, ν' satisfying $\bar{\nu} > \nu > \nu' > 0$, $|\alpha^\nu(w,s) - \alpha^{\nu'}(w,s)| < \varepsilon$ for almost all s and all w.

For any monotone strategy profile α let $S_d(\alpha)$ be the set of signals at which $\alpha(w,\cdot)$ is discontinuous for a positive measure of types w. Since $\alpha(w,\cdot)$ is monotone, $S_d(\alpha)$ is at most countable, i.e., it has zero measure for all $\nu > 0$. Recall that, conditional on an agent receiving s, the profile of realized signals must lie inside $[s-\nu, s+\nu]$. Hence, given that signals and types are continuously distributed, the mass of agents with signals that belong to $S_d(\alpha)$ conditional on a player receiving signal $s \notin S_d(\alpha)$ must be zero for all $\nu \geq 0$, that is, even in the limit since $[s-\nu, s+\nu] \to \{s\}$ as $\nu \to 0$. But this implies that the aggregate action $\bar{\alpha}^\nu(\alpha;\theta)$ given by (28) converges uniformly to

$$\bar{\alpha}(\alpha; s) := \int_{\underline{w}}^{\overline{w}} \alpha(w, s) dF(w).$$

To see why, notice that, since $\theta \in [s - \nu/2, s + \nu/2]$, by the monotonicity of α we can bound $\bar{\alpha}^{\nu}(\alpha; \theta)$ as follows:

$$\int_{\underline{w}}^{\overline{w}} \int_{-1/2}^{1/2} \alpha(w, s - \nu/2 + \nu\eta) h_w(\eta) d\eta dF(w) \leq \bar{\alpha}^{\nu}(\alpha; \theta) \leq \int_{\underline{w}}^{\overline{w}} \int_{-1/2}^{1/2} \alpha(w, s + \nu/2 + \nu\eta) h_w(\eta) d\eta dF(w).$$

Since $s \pm \nu/2 + \nu\eta \in [s - \nu, s + \nu]$ for all η and $s \notin S_d(\alpha)$ the set of pairs $(w, s \pm \nu/2 + \nu\eta)$ at which α is discontinuous has zero measure so both bounds must converge to $\bar{\alpha}(\alpha; s)$. Moreover, $\alpha(w, s - \nu/2 + \nu\eta)$ and $\alpha(w, s + \nu/2 + \nu\eta)$ respectively increase and decrease as $\nu \to 0$ for all $\eta \in (-1/2, 1/2)$, implying that the lower bound monotonically increases and the upper bound monotonically decreases. That is, the convergence of $\bar{\alpha}^{\nu}(\alpha; \theta)$ to $\bar{\alpha}(\alpha, s)$ must be uniform.

In turn, by the Lipschitz continuity of ΔU, the uniform convergence of $\bar{\alpha}^{\nu}(\alpha; \theta)$ implies that expected payoff differences between any two actions conditional on receiving $s \notin S_d(\alpha)$ uniformly converge as $\nu \to 0$ for any fixed monotone profile α. That is, for all $s \notin S_d(\alpha)$ and all w and all $\varepsilon > 0$, there is $\hat{\nu} > 0$ such that

$$|E_{\nu}[\Delta U(\alpha(w, s), a', \bar{\alpha}^{\nu}(\alpha; \theta), \theta, w)|s] - E_{\nu'}[\Delta U(\alpha(w, s), a', \bar{\alpha}^{\nu'}(\alpha, \theta), \theta, w)|s]| < \varepsilon$$

for all $\nu, \nu' < \hat{\nu}$, where E_{ν} denotes the expectation operator under noise level ν.

Equipped with this result we next argue that the equilibrium profiles must converge. We do so by showing that for all $\nu, \nu' < \hat{\nu}$ the largest signal shift needed to make α^{ν} equal to $\alpha^{\nu'}$ is $O(\varepsilon)$. To account for discontinuities in $\alpha^{\nu'}$, define such a signal shift as

$$\hat{\delta} = \max \Big\{ \delta : \alpha^{\nu}(w, s + \delta) \in [\liminf \alpha^{\nu'}(s, w), \limsup \alpha^{\nu'}(s, w)] \\ \text{for some } (w, s) \text{ s.t. } \alpha^{\nu}(w, s + \delta) \neq \alpha^{\nu'}(w, s + \delta) \Big\}.$$

Assume that $\hat{\delta} > 0$ (otherwise we can apply a similar argument by switching α^{ν} and $\alpha^{\nu'}$). Note that $\alpha^{\nu'}(w, s - \hat{\delta}) \leq \alpha^{\nu}(w, s)$ for all s, w. Fix signal-type pair $(\hat{w}, \hat{s})$ associated with the signal shift $\hat{\delta}$, implying that $\alpha^{\nu'}(w, s) > \alpha^{\nu}(w, s)$. By Lemma 8 and the fact that $\alpha^{\nu}, \alpha^{\nu'}$ are respectively the equilibrium strategies under noise levels

ν, ν', we have that

$$\begin{aligned}
0 &\leq E_{\nu'}[\Delta U(\alpha^{\nu'}(\hat{w}, \hat{s}), \alpha^{\nu}(\hat{w}, \hat{s}), \bar{\alpha}(\alpha^{\nu'}), \theta, \hat{w})|\hat{s}] \\
&\leq E_{\nu'}[\Delta U(\alpha^{\nu'}(\hat{w}, \hat{s}), \alpha^{\nu}(\hat{w}, \hat{s}), \bar{\alpha}(\alpha^{\nu'}_{\hat{\delta}}), \theta, \hat{w})|\hat{s} + \hat{\delta}] - K(\alpha^{\nu'}(\hat{w}, \hat{s}) - \alpha^{\nu}(\hat{w}, \hat{s}))\hat{\delta} \\
&\leq E_{\nu'}[\Delta U(\alpha^{\nu'}(\hat{w}, \hat{s}), \alpha^{\nu}(\hat{w}, \hat{s}), \bar{\alpha}(\alpha^{\nu}), \theta, \hat{w})|\hat{s} + \hat{\delta}] - K(\alpha^{\nu'}(\hat{w}, \hat{s}) - \alpha^{\nu}(\hat{w}, \hat{s}))\hat{\delta} \\
&\leq E_{\nu}[\Delta U(\alpha^{\nu'}(\hat{w}, \hat{s}), \alpha^{\nu}(\hat{w}, \hat{s}), \bar{\alpha}(\alpha^{\nu}), \theta, \hat{w})|\hat{s} + \hat{\delta}] + \varepsilon - K(\alpha^{\nu'}(\hat{w}, \hat{s}) - \alpha^{\nu}(\hat{w}, \hat{s}))\hat{\delta} \\
&\leq E_{\nu}[\Delta U(\alpha^{\nu'}(\hat{w}, \hat{s}), \alpha^{\nu}(\hat{w}, \hat{s} + \hat{\delta}), \bar{\alpha}(\alpha^{\nu}), \theta, \hat{w})|\hat{s} + \hat{\delta}] + \varepsilon - K(\alpha^{\nu'}(\hat{w}, \hat{s}) - \alpha^{\nu}(\hat{w}, \hat{s}))\hat{\delta} \\
&\leq \varepsilon - K(\alpha^{\nu'}(\hat{w}, \hat{s}) - \alpha^{\nu}(\hat{w}, \hat{s}))\hat{\delta}.
\end{aligned}$$

Since $\alpha^{\nu'}(\hat{w}, \hat{s}) - \alpha^{\nu}(\hat{w}, \hat{s})$ is bounded by a_{max} then $\hat{\delta} \leq \frac{\varepsilon}{Ka_{max}}$. That is, equilibrium strategies converge to some limit profile $\alpha_G(w, s)$ for almost all s and w.

Finally, notice that the convergence of equilibrium strategies and the uniform convergence of expected payoff differences, combined with the fact that $s \to \theta$ as $\nu \to 0$, imply that

$$\begin{aligned}
\lim_{\nu \to 0} E_{\nu}[\Delta U(\alpha^{\nu}(w, s), a, \bar{\alpha}^{\nu}(\alpha^{\nu}; \theta), \theta, w)|s] &= E[\Delta U(\alpha_G(w, s), a, \bar{\alpha}(\alpha_G(\cdot, s)), \theta, w)|s = \theta] \\
&= \Delta U(\alpha_G(w, \theta), a, \bar{\alpha}(\bar{\alpha}(\alpha_G(\cdot, \theta)), \theta, w),
\end{aligned}$$

for almost all s. Since α^{ν} is an equilibrium on Γ^{ν}_{θ}, we have that

$$E_{\nu}(\Delta U(\alpha^{\nu}(w, s), a, \bar{\alpha}^{\nu}(\alpha^{\nu}; \theta), \theta, w)|s) \geq 0,$$

for all s and all w. By the continuity of ΔU the above convergence implies that

$$\Delta U(\alpha_G(w, \theta), a, \bar{\alpha}(\alpha_G(\cdot, \theta)), \theta, w) \geq 0,$$

for almost all θ and all w. That is, α_G is a NE of Γ_{θ} for almost all θ. By Lemma 7 the result extends to the non-uniform prior case. □

B.4 Proofs of Results in Subsection 4.3

Proof of Theorem 1. To prove the equivalence between α_P and α_G for almost all (w, θ), we need to focus on the case when Γ_{θ} has multiple NE that differ in a positive measure set of types but it has an essentially unique potential maximizer. This is because (i) when Γ_{θ} has an essentially unique NE it must coincide a.e. with both $\alpha_P(\cdot, \theta)$ and $\alpha_G(\cdot, \theta)$ by Propositions 3 and 5; and (ii) these propositions also imply that the set of θ at which Γ_{θ} has multiple potential maximizers that differ in a positive measure set of types or at which the global games selection is not essentially unique has measure zero.

Focus first on the uniform-prior case. The proof applies the Generalized Laplacian property (Lemma 3) to show that if $\alpha_P(\cdot,\theta)$ is the essentially unique potential maximizer of Γ_θ then

$$\lim_{\nu\to 0} E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}^\nu(\alpha_P;\theta),\theta,w)|s]>0$$

for all $a\neq\alpha_P(w,\theta)$, all $s\in(\inf\Theta+\nu/2,\sup\Theta-\nu/2)$ and almost all w, where $\bar{\alpha}^\nu(\alpha_P;\theta)$ represents the aggregate action given θ when agents follow strategies $\alpha_P(w,s)$. But, as shown in the proof of Proposition 5, these conditions fully characterize the limit equilibrium in the global game, which is essentially unique by Proposition 4. Hence, it must be that $\alpha_P(w,\theta)=\alpha_G(w,\theta)$ for almost all w. Since, the potential maximizer is essentially unique for almost all θ then α_G and α_P must be equal except possibly in a measure zero set of (w,θ).

Fix any $\theta\in(\inf\Theta,\sup\Theta)$ for which there is an essentially unique potential maximizer. For any given action $a\in A$ and any subset of types W with positive measure such that $\alpha_P(w,\theta)>a$ for all $w\in W$, let $\alpha(w)=a$ if $w\in W$ and $\alpha(w)=\alpha_P(w,\theta)$ if $w\notin W$. Since α_P is the unique potential maximizer, we have that

$$V(\alpha_P,\theta)-V(\alpha,\theta)=\int_{\bar{\alpha}(\alpha)}^{\bar{\alpha}(\alpha_P(\cdot,\theta))}u(z,\theta)dz+\int_W\left(v(\alpha_P(w,\theta),\theta,w)-v(a,\theta,w)\right)dF(w)>0.$$

Let $\alpha(w,s)$ represent the monotone strategy in the global game in which agents of types in W switch from a to $\alpha_P(w,\theta)$ using cutoff function $\kappa(w)=\theta$ for all $w\in W$; while types in $W^C:=[\underline{w},\overline{w}]\setminus W$ choose $\alpha(w,s)=\alpha_P(w,\theta)$ for all s. We can express their expected payoff differences conditional *both* on $s=\kappa(w)$ *and* on θ as follows:

$$\begin{aligned}&E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}^\nu(\alpha;\theta),\theta,w)|s=\theta,\theta]\\&=\int_a^{\bar{\alpha}(\alpha_P(\cdot,\theta),W)}(\alpha_P(w,\theta)-a)u\big(\bar{\alpha}(\alpha_P(\cdot,\theta),W^C)F(W^C)+zF(W),\theta\big)dG_w(z|\kappa;\alpha,W)\\&+\left(v(\alpha_P(w,\theta),\theta,w)-v(a,\theta,w)\right),\end{aligned}$$

where z represents the aggregate action of agents with types in W and $F(W^C),F(W)$ respectively denote the probability mass of W^C and W. Integrating the above expression over types in W and applying Lemma 3 we obtain

$$\int_W E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}^\nu(\alpha;\theta),\theta,w)|s=\theta,\theta]dF(w|w\in W)$$

$$= \int_a^{\bar{\alpha}(\alpha_P(\cdot,\theta),W)} u\big(\bar{\alpha}(\alpha_P(\cdot,\theta),W^C)F(W^C)+zF(W),\theta\big)dz$$

$$+\int_W (v(\alpha_P(w,\theta),\theta,w)-v(a,\theta,w))\,dF(w|w\in W). \tag{31}$$

Applying the change of variable $z' = \bar{\alpha}(\alpha_P(\theta,\cdot),W^C)F(W^C)+zF(W)$ and since $dF(w|w\in W) = dF(w)/F(W)$ we have that

$$\int_W E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}^\nu(\alpha;\theta),\theta,w)|s=\theta,\theta]dF(w|w\in W)$$

$$= \int_{\bar{\alpha}(\alpha)}^{\bar{\alpha}(\alpha_P(\cdot,\theta))} u(z',\theta)\frac{dz'}{F(W)} + \int_W (v(\alpha_P(w,\theta),\theta,w)-v(a,\theta,w))\,\frac{dF(w)}{F(W)}$$

$$= \frac{1}{F(W)}\big(V(\alpha_P,\theta)-V(\alpha,\theta)\big) > 0. \tag{32}$$

Since $\alpha_P(w,\theta)\geq\alpha(w)$ for all w, with strict inequality for all $w\in W$, we have that $\bar{\alpha}(\alpha_P(\cdot,\theta)) > \bar{\alpha}^\nu(\alpha;\theta)$. Hence, by strict increasing differences we have that

$$\begin{aligned}E_\nu[\Delta U(\alpha_P(w,s),a,&\bar{\alpha}(\alpha_P(\cdot,\theta)),\theta,w)|s=\theta,\theta]\\ &> E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}^\nu(\alpha;\theta),\theta,w)|s=\theta,\theta].\end{aligned}$$

Also, as $\nu\to 0$ expected payoff differences conditional on $s=\theta$ and θ converge to the expected payoff differences conditional on only $s=\theta$ since $s\to\theta$. Accordingly,

$$\lim_{\nu\to 0}\int_W E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}(\alpha_P(\cdot,\theta)),\theta,w)|s=\theta]dF(w|w\in W) > 0. \tag{33}$$

A symmetric argument shows that (33) for any $a>\alpha_P(w,\theta)$. Hence, inequality (33) is satisfied for any a and any subset W such that either $a<\alpha_P(w,\theta)$ or $a>\alpha_P(w,\theta)$, implying that

$$\lim_{\nu\to 0} E_\nu[\Delta U(\alpha_P(w,s),a,\bar{\alpha}(\alpha_P(\cdot,\theta)),\theta,w)|s=\theta] > 0$$

for almost all w and all $a\neq\alpha_P(w,\theta)$. Since $\alpha_P(w,\cdot)$ is monotone it is continuous a.e., implying that the set of θ at which a positive mass of agents have a discontinuity in their strategies has measure zero. Hence, we have that $\lim_{\nu\to 0}\bar{\alpha}^{\nu}(\alpha_P;\theta) = \bar{\alpha}(\alpha_P(\cdot,\theta))$ for almost all $\theta \in (\inf\Theta, \sup\Theta)$, and the convergence is uniform. By the Lipschitz continuity of ΔU, this implies that

$$\lim_{\nu\to 0} E_{\nu}[\Delta U(\alpha_P(w,s), a, \bar{\alpha}^{\nu}(\alpha_P;\theta)), \theta, w)|s] > 0$$

for almost all w and all $a \neq \alpha_P(w,\theta)$.

Finally, to prove the proposition under a general prior it suffices to show that (31) and (32) hold in the limit. This is because, in such a case, all the above arguments continue to apply when the prior is not uniform.

Note that the signal cutoff function is set to be $\kappa(w) = \theta$ for all W. Given such a cutoff function, we have that

$$\frac{\phi(\kappa(w))f(w)}{\int_W \phi(\kappa(w))f(w)dw} = \frac{f(w)}{\int_W f(w)dw} = f(w|w\in W).$$

Hence, applying Lemma 5 with $\kappa(w) = \theta$ for all W and taking the limit as $\nu \to 0$ we obtain (31) and (32). $\square$

B.5 Proofs of Results in Section 5

Proof of Lemma 2. We prove the existence and continuity of $B(\bar{a}, v)$ by first showing that a solution to the maximization problem exists if we restrict attention to step functions and then applying Lemma 6 to argue that any measurable function in $\mathcal{A}_{\bar{a}}$ can be approximated by a sequence of step functions.

Let $\mathcal{A}^n_{\bar{a},\varepsilon}$ denote the set of step functions in $\mathcal{A}^n$ with aggregate action within ε of $\bar{a}$. That is,

$$\mathcal{A}^n_{\bar{a},\varepsilon} = \{\alpha \in \mathcal{A}^n : |\bar{\alpha}(\alpha) - \bar{a}| \leq \varepsilon\}.$$

$\mathcal{A}^n_{\bar{a},\varepsilon}$ is compact and non-empty for large enough n. This is because if the interval partition $\{W^n_j\}^n_{j=1}$ of the type space is fine enough then, by the continuity of F, we can find a vector of actions $(a_1, \cdots, a_n) \in A^n$ that yields an aggregate action within ε of $\bar{a}$. Hence,

$$\max_{\alpha\in\mathcal{A}^n_{\bar{a},\varepsilon}} \int_w v(\alpha(w), \theta, w)dF(w)$$

has a solution. Moreover, since the objective function is bounded and $\mathcal{A}^{2^n}_{\bar{a},\varepsilon} \subset \mathcal{A}^{2^{(n+1)}}_{\bar{a},\varepsilon}$ the value function associated with the sequence of constraint sets $\{\mathcal{A}^{2^n}_{\bar{a},\varepsilon}\}$ is increasing in n and bounded so it converges for all $\bar{a}$ and $\varepsilon > 0$.

Next, note that, by Lemma 6 and the continuity of v and F, for any $\alpha \in \mathcal{A}_{\bar{a}}$ and any $\varepsilon > 0$ we can find a sequence $\{\alpha^n\}$ with $\alpha^n \in \mathcal{A}^n_{\bar{a},\varepsilon}$ such that

$$\lim_{n\to\infty} \int_w v(\alpha^n(w), \theta, w)dF(w) = \int_w v(\alpha(w), \theta, w)dF(w).$$

Accordingly,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\max_{\alpha\in\mathcal{A}^{2^n}_{\bar{a},\varepsilon}}\int_w v(\alpha(w),\theta,w)dF(w)=\max_{\alpha\in\mathcal{A}_{\bar{a}}}\int_w v(\alpha(w),\theta,w)dF(w):=B(\bar{a},\theta).$$

That is, $B(\bar{a},\theta)$ is well-defined given that the limit in the left hand side exists. We use a similar argument to show the continuity of $B(\cdot,\theta)$. Note that, for any $\varepsilon'<\varepsilon$ and any $\alpha'\in\mathcal{A}_{\bar{a}+\varepsilon'}$, we can find a sequence $\{\alpha'^n\}$ with $\alpha'^n\in\mathcal{A}^n_{\bar{a},\varepsilon}$ such that

$$\lim_{n\to\infty}\int_w v(\alpha'^n(w),\theta,w)dF(w)=\int_w v(\alpha'(w),\theta,w)dF(w)$$

and $\lim_{n\to\infty}\bar{\alpha}(\alpha'^n)=\bar{a}+\varepsilon'$. Letting $\varepsilon=2\varepsilon'$ we get that

$$\begin{aligned}\lim_{\varepsilon'\to 0}\max_{\alpha'\in\mathcal{A}_{\bar{a}+\varepsilon'}}\int_w v(\alpha(w),\theta,w)dF(w)&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\max_{\alpha\in\mathcal{A}^{2^n}_{\bar{a},2\varepsilon}}\int_w v(\alpha(w),\theta,w)dF(w)\\&=\max_{\alpha\in\mathcal{A}_{\bar{a}}}\int_w v(\alpha(w),\theta,w)dF(w)\end{aligned}$$

We next show that NE aggregate actions must satisfy (14). If α^* is a NE with aggregate action $\bar{a}^*$ then, for all a and w

$$\alpha^*(w)u(\bar{a}^*,\theta)+v(\alpha^*(w),\theta,w)\geq au(\bar{a}^*,\theta)+v(a,\theta,w).$$

Integrating these conditions across types we get that, for any strategy profile $\alpha\in\mathcal{A}$,

$$\bar{a}^*u(\bar{a}^*,\theta)+\int_w v(\alpha^*(w),\theta,w)dF(w)\geq\bar{\alpha}(\alpha)u(\bar{a}^*,\theta)+\int_w v(\alpha(w),\theta,w)dF(w).\quad(34)$$

But notice that, for any given aggregate action $\bar{a}\in[0,a_{max}]$ and any $\alpha\in\mathcal{A}_{\bar{a}}$, we have that $\int_w\alpha(w)u(\bar{a},\theta)dF(w)=\bar{a}u(\bar{a},\theta)$. Hence, by applying inequality (34) to the set of strategy profiles $\mathcal{A}_{\bar{a}^*}$ we obtain

$$\alpha^*\in\arg\max_{\alpha\in\mathcal{A}_{\bar{a}^*}}\int_w v(\alpha(w),\theta,w)dF(w).\quad(35)$$

That is, NE α^* yields an average idiosyncratic payoff equal to $B(\bar{a}^*,\theta)$. Accordingly, (34) implies that the NE aggregate action solved the following fixed point problem:

$$\bar{a}^*\in\arg\max_{\bar{a}\in[0,a_{max}]}\big(\bar{a}u(\bar{a}^*,\theta)+B(\bar{a},\theta)\big).$$

This proves the "if" part. To prove the "only if" part notice that if α^* satisfies (35) then it must yield the same average payoff as any NE profile associated with aggregate action $\bar{a}^*$. But then individual payoffs given $\bar{a}^*$ must be maximized for all but a subset of types with measure zero, otherwise such NE would violate (34), and hence, the aggregate action of the two profiles must coincide. $\square$

Proof of Theorem 2. First, note that the objective function in (14) is continuous in $\bar{a}$. Next, if $u(\cdot, \theta)$ is strictly increasing and $B(\cdot, \cdot)$ exhibits strictly increasing differences then, for any fixed $\bar{a}^*$, the objective function in (14) also has strictly increasing differences in $\bar{a}$ and θ. Finally, $[0, a_{\max}]$ is compact.

Accordingly, by Topkis monotonicity theorem the correspondence

$$m(\bar{a}^*, \theta) := \arg\max_{\bar{a} \in [0, a_{\max}]} \big(\bar{a} u(\bar{a}^*, \theta) + B(\bar{a}, \theta)\big)$$

is increasing in θ in the strong set order. In addition, $m(\bar{a}^*, \theta)$ is non-empty and upper hemicontinuous by Berge's maximum theorem.

Finally, notice that $\min m(0, \theta) \geq 0$ and $\max m(a_{max}, \theta) \leq a_{max}$ for all θ. That is, the correspondence $m(a_{max})$ 'crosses' the diagonal from above at the smallest and largest crossing values of $\bar{a}^*$. But since the NE aggregate actions satisfy $\bar{a}^* \in m(\bar{a}^*, \theta)$, i.e., they are represented by the values at which $m(\cdot, \theta)$ crosses the diagonal, the smallest and largest crossing values must go up as $m(\cdot, \theta)$ increases with θ. $\square$

Proof of Theorem 3. Given any $\alpha \in int(\mathcal{A})$, consider an infinitesimal increase da of all individual actions $\alpha(w)$. Since v is Lipschitz continuous it is differentiable almost everywhere, the change in average idiosyncratic payoffs for almost all $\alpha \in int(\mathcal{A})$ is given by

$$\int_{\underline{w}}^{\overline{w}} \frac{\partial v(a, \theta, w)}{\partial a}\Big|_{a=\alpha(w)} dF(w).$$

But notice that such an increase in individual actions implies that the aggregate action also increases by da. Hence, we must have that

$$\frac{\partial B(\bar{a}, \theta)}{\partial \bar{a}} \geq \max_{\alpha \in int(\mathcal{A}_{\bar{a}})} \int_{\underline{w}}^{\overline{w}} \frac{\partial v(a, \theta, w)}{\partial a}\Big|_{a=\alpha(w)} dF(w),$$

given that we can always find a new profile $\alpha' \in \mathcal{A}_{\bar{a}+da}$ that yields a weakly higher average idiosyncratic payoff than the profile $\alpha(w) + da$ for all w, which also belongs to $\in \mathcal{A}_{\bar{a}+da}$. But then, since $\int_{\underline{w}}^{\overline{w}} \frac{\partial^2 v(a,\theta,w)}{\partial a \partial \theta}\Big|_{a=\alpha(w)} dF(w) > 0$ for all $\alpha \in int(\mathcal{A}_{\bar{a}})$ we can apply a similar argument to argue that $\frac{\partial B(\bar{a},\theta)}{\partial \bar{a}}$ must be increasing in θ: we can always find $\alpha'' \in \mathcal{A}_{\bar{a}+da}$ that leads to a further increase in average idiosyncratic payoffs over those associated with α', i.e.,

$$\frac{\partial^2 B(\bar{a}, \theta)}{\partial \bar{a} \partial \theta} \geq \max_{\alpha \in int(\mathcal{A}_{\bar{a}})} \int_{\underline{w}}^{\overline{w}} \frac{\partial^2 v(a, \theta, w)}{\partial a \partial \theta}\Big|_{a=\alpha(w)} dF(w) > 0.$$

That is, $B(\bar{a}, \theta)$ exhibits strictly increasing differences in $(0, a_{max}) \times \Theta'$ and, by continuity, in $[0, a_{max}] \times \Theta'$ so Theorem 2 applies. $\square$

Proof of Theorem 4. The proposition immediately follows by noting that (i) the first term in the objective function of (14) does not depend on F and, (ii) we can index F and $\hat{F}$ using parameter $\zeta \in \mathbb{R}$ so that the condition in the proposition is equivalent to B exhibiting increasing differences in $\bar{a}$ and ζ. Together, (i) and (ii) imply that u and B satisfy the conditions in Theorem 2 w.r.t. $\bar{a}$ and ζ (keeping θ unchanged). $\square$

Proof of Theorem 5. The proof logic is similar to the one used in Theorem 3 and is therefore omitted. $\square$

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