August 28, 2018

ONLINE APPENDIX

Financial Contracting with Enforcement Externalities

Lukasz A. Drozd and Ricardo Serrano-Padial

Contents

1 Strategic Complementarities: Numerical Illustration

To highlight the impact of strategic complementarities on the provision of credit implied by the selected value of rents from lack of enforcement γ, Figure A1 compares credit provision under two scenarios. The first scenario assumes that, whenever there are multiple equilibria under common knowledge of X, the low-default equilibrium is always played ('good eq.'). The second one assumes that the highest-default equilibrium is played ('bad eq.'). As the figure shows, when $\gamma = 0.33$, the differences in credit supply range from 5 percent to 20 percent. As complementarities become stronger due to higher rents from the lack of enforcement, differences in credit go up substantially (they are in the 20-40% range for $\gamma = 0.5$). The figure also shows that the link between X and b underlying Proposition 6 is robust to the strength of complementarities, since in both the good and bad equilibrium scenarios, and for both values of γ, credit increases with X.

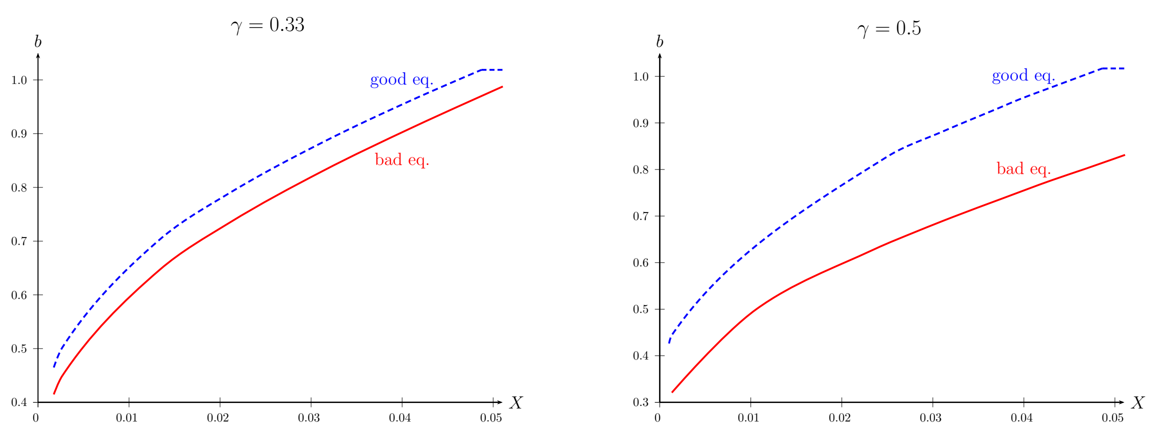

Figure A1: Credit under Common Knowledge of X.

Notes: The figure compares equilibrium credit provision assuming two alternative scenarios. Whenever there are multiple equilibria under common knowledge of X, i) the low-default equilibrium is always played ('good eq.'), and ii) the highest-default equilibrium is played ('bad eq.').

2 Omitted Proofs

Lemma 7 (belief constraint)**.** *For any subset $W' \subseteq \mathcal{W}^*$ and any $z \in [0,1]$,*

$$\frac{1}{\sum_{W'} f(w)} \sum_{W'} Pr\left(\psi(X, W') \leq z \middle| x = k(w)\right) f(w) = z, \tag{A1}$$

where $Pr\left(\cdot \middle| x = k(w)\right)$ is the probability assessment of $\psi(W', X)$ by an agent receiving $x = k(w)$.

Proof (based on Sákovics and Steiner, 2012). The proof of the belief constraint is given by the following steps.

First, we define "virtual signals" $\tilde{x} = x - k(w)$ for all $w \in W'$, which are a function of the random vector (x, w), which represents the type of a player. Virtual signals exhibit a common default threshold $\tilde{k} = 0$.

Second, we show below that, given the uniform prior on X, the probability that $x = k(w)$ for a type (x, w), conditional on the virtual threshold signal ($\tilde{x} = 0$), is given by the fraction of returns w in subset W', i.e.,

$$Pr(k(w), w | \tilde{x} = 0) = \frac{f(w)}{\sum_{W'} f(w)}, \tag{A2}$$

where $Pr(x, w|\cdot)$ denotes the conditional probability density of type (x, w).

Third, we also show below that the default rate $\psi(X, W')$, conditional on $\tilde{x} = 0$, is uniformly distributed in $[0, 1]$. That is,

$$Pr(\psi(X, W') < z | \tilde{x} = 0) = z. \tag{A3}$$

This is due to the fact the "virtual noise" associated to $\tilde{x}$, defined as $\tilde{\eta} = (\tilde{x} - X)/\nu$, is i.i.d.

and, hence, the aggregate action in W' satisfies the *Laplacian property* (Morris and Shin, 2003).

Finally, combining (A2) and (A3), we have that

$$z = Pr(\psi(X,W') < z|\tilde{x}=0) = \sum_{W'} Pr\left(\psi(X,W') \leq z \middle| x = k(w)\right) Pr(x = k(w)|\tilde{x}=0)$$
$$= \frac{1}{\sum_{W'} f(w)} \sum_{W'} Pr\left(\psi(X,W') \leq z \middle| x = k(w)\right) f(w).$$

To prove (A2) we need to find the marginal distributions of (x,w) and $\tilde{x}$. First, recall that threshold signals fall in $[\nu/2, 1-\nu/2]$, as we have shown in the proof of Lemma 6. Hence, we need to focus only on the distribution of signals $x \in [\nu/2, 1-\nu/2]$. Next notice that, since X, ν and w are independent and $X \sim U[0,1]$, the joint density of (x,w,X) is given by

$$Pr(x,w,X) = Pr(x|w,X)Pr(w|X)Pr(X) = h\left(\frac{x-X}{\nu}\right)\frac{1}{\nu}\frac{f(w)}{\sum_{W'} f(w)},$$

where h denotes the density of noise η. Given this, the marginal density of (x,w) is

$$Pr(x,w) = \int_{x-\nu/2}^{x+\nu/2} Pr(x,w,X)dX = \int_{x-\nu/2}^{x+\nu/2} h\left(\frac{x-X}{\nu}\right)\frac{1}{\nu}\frac{f(w)}{\sum_{W'} f(w)}dX = \frac{f(w)}{\sum_{W'} f(w)}.$$

The marginal density of the virtual signal $\tilde{x} = x - k(w)$ is given by

$$Pr(x = \tilde{x} + k(w)) = \sum_{W'} Pr(\tilde{x} + k(w), w) = 1,$$

for all $\tilde{x}$ such that $\tilde{x} + k(w) \in [\nu/2, 1-\nu/2]$. Since $\tilde{x} = 0$ satisfies this condition we have that

$$Pr(k(w), w|\tilde{x}=0) = \frac{Pr(k(w),w)}{Pr(k(w))} = \frac{f(w)}{\sum_{W'} f(w)}.$$

To prove (A3) notice that virtual noise $\tilde{\eta} = (\tilde{x} - X)/\nu$ is drawn from the mixture distribution $\left\{H\left(\tilde{\eta} + \frac{k(w)}{\nu}\right), \frac{f(w)}{\sum_{W'} f(w)}\right\}_{w \in W'}$, meaning that (i) with probability $\frac{f(w)}{\sum_{W'} f(w)}$ the virtual noise belongs to type w and, conditional on type w, (ii) its distribution is given by the noise distribution evaluated at $\eta = \tilde{\eta} + k(w)/\nu$., i.e., at the noise level associated to $\tilde{\eta}$. This mixture distribution does not depend on X and thus $\tilde{\eta}$ is i.i.d. across agents and independent of X. Denote by G the (continuous) cdf of $\tilde{\eta}$ and let $G^{-1}(z) = \inf\{\tilde{\eta} : G(\tilde{\eta}) = z\}$. The default rate $\psi(X, W')$ is given by the fraction of agents in W' whose virtual signal is lower than zero, i.e., by those with virtual noise below $-X/\nu$. Accordingly,

$$
\begin{aligned}
Pr(\psi(X, W') < z | \tilde{x} = 0) &= Pr(G(-X/\nu) < z | \tilde{x} = 0) = Pr(G(\tilde{\eta}) < z) \\
&= Pr(\tilde{\eta} < G^{-1}(z)) = G(G^{-1}(z)) = z.
\end{aligned}
$$

□

Lemma 8. *There exist a unique partition* $\{W_1, \cdots, W_I\}$ *and a set of thresholds* $k_1 > k_2 > \cdots > k_I$ *such that, as* $\nu \to 0$*, for all* $w \in W_i$*,* $i = 1, \cdots, I$*,* $k^\nu(w)$ *uniformly converges to* k_i*. Moreover, thresholds* $\mathbf{k} = (k_1, \cdots, k_I)$ *solve the system of limit indifference conditions*

$$
\int_0^1 P\left(k_i, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w')\right) dA_w(z|\mathbf{k}, W_i) = \theta(w), \quad \forall w \in W_i,\ \forall i, \tag{A4}
$$

where $A_w(z|\mathbf{k}, W_i)$ *represents the strategic beliefs of type-*w *agents in the limit and satisfies the belief constraint (22).*

Proof. To prove convergence, we first partition the set of types into subsets W_i of types for sufficiently small ν as follows: (i) if we order the signal thresholds of all types, any adjacent thresholds that are within ν of each other belong to the same subset, and (ii) $j > i$ implies that the thresholds associated with types in W_j are lower than those associated with W_i by at least ν. Also, let $Q_w^\nu(\mathrm{x}|\mathbf{k}^\nu, z) := Pr\left(X \leq \mathrm{x}\middle|x = k^\nu(w), \psi(X, W_i) = z\right)$ represent the beliefs about capacity X of an agent of type $w \in W_i$ conditional on receiving her threshold signal $k^\nu(w)$ and on the event that the default rate in W_i is equal to z.

Note that a type-w agent receiving signal $x = k^\nu(w)$ knows that all agents with types in W_j are defaulting if $j < i$ and repaying if $j > i$. Also, the support of $Q_w^\nu(\cdot|\mathbf{k}^\nu, z)$ must lie within $[k^\nu(w) - \nu/2, k^\nu(w) + \nu/2]$. Given this, by the law of iterated expectations, her expected enforcement probability conditional on $x = k^\nu(w)$ can be written in terms of her strategic belief as follows:

$$E(P|\mathbf{k}^\nu; k^\nu(w)) =$$

$$\int_0^1 \int_{k^\nu(w)-\nu/2}^{k^\nu(w)+\nu/2} P\left(\mathrm{x}, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w')\right) dQ_w^\nu(\mathrm{x}|\mathbf{k}^\nu, z) dA_w(z|\mathbf{k}^\nu, W_i). \quad \text{(A5)}$$

In addition, notice that we can always express $E(P|\mathbf{k}^\nu; k^\nu(w))$ in terms of the threshold signal $k^\nu(w)$ and relative threshold differences $\Delta_{w'} = (k^\nu(w') - k^\nu(w))/\nu$. Importantly, as Sákovics and Steiner (2012) emphasize, strategic beliefs depend on the relative distance between thresholds $\Delta_{W_i} = \{\Delta_{w'}\}_{w' \in W_i}$ rather than on their absolute distance. That is, keeping Δ_{W_i} fixed, $A_w(z|\mathbf{k}^\nu, W_i)$ does not change with ν.[1] This implies that strategic beliefs satisfy the belief constraint when $\nu = 0$.

Fix $k^\nu(w) = k_i$ for some $w \in W_i$ and fix Δ_{W_i}, for all $i = 1, \cdots, I$ and all ν sufficiently small. By fixing relative differences, the partition $\{W_i\}_{i=1}^I$ still satisfies the above definition and thus, does not change as $\nu \to 0$. We are going to show that indifference condition

[1] This is straightforward to check. First, if we substitute $X = k^\nu(w) - \nu\eta$ (since agents with type w get her threshold signal) and $k(w') = \nu\Delta_{w'} + k^\nu(w)$ into (21), we find that $\psi(X, W_i)$ only depends on Δ_{W_i} and $k^\nu(w)$. But this means that $A_w(z|\mathbf{k}^\nu, W_i)$ only depends on Δ_{W_i} and $k^\nu(w)$ because h is independent of ν.

$E(P|\mathbf{k}^\nu; k^\nu(w)) = \theta(w)$ is approximated by the limit condition in the lemma for ν sufficiently small.

Note that the inner integral in (A5) is bounded below by

$$P\left(k^\nu(w) - \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w')\right)$$

and above by

$$P\left(k^\nu(w) + \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w')\right).$$

Hence,

$$\int_0^1 P\left(k_i - \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w')\right) dA_w(z|\mathbf{k}^\nu, W_i) \leq E(P|\mathbf{k}^\nu; k^\nu(w))$$
$$\leq \int_0^1 P\left(k_i + \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w')\right) dA_w(z|\mathbf{k}^\nu, W_i). \quad \text{(A6)}$$

The first term inside these integrals is Lipschitz continuous by Assumption 1. In addition, the next lemma shows that $dA_w(z|\mathbf{k}^\nu, k^\nu(w))$ is bounded for all ν.

Lemma 10. $0 \leq \dfrac{\partial A_w(z|\mathbf{k}^\nu, k^\nu(w))}{\partial z} \leq \dfrac{\sum_{W_i} f(w')}{f(w)}$ *for all* $w \in W_i$ *and all* z *in the support of* $A_w(\cdot|\mathbf{k}^\nu, k^\nu(w))$.

See proof below.

Hence, the LHS and the RHS of (A6) uniformly converge to each other as $\nu \to 0$, leading to limit indifference conditions (A4). Note also that $k^\nu(w) \in [-\bar{\nu}/2, 1 + \bar{\nu}/2]$ and, keeping $\{W_i\}_{i=1}^I$ fixed, $\Delta_{w'}$ are uniformly bounded for all $w' \in W_i$ since any two adjacent thresholds of types in W_i are within ν of each other. That is, the solution to the system of indifference conditions $E(P|\mathbf{k}^\nu; k^\nu(w)) = \theta(w)$ lies in a compact set.[2] Accordingly, we can find $\hat{\nu}$ so that indifference conditions are within ε of the limit condition for all $\nu < \hat{\nu}$, leading to their solutions being in a neighborhood of the solution $\mathbf{k}$ of limit indifference conditions (A4). □

Proof of Lemma 10. Let $\psi^{-1}(z, W_i)$ be the inverse function of $\psi(X, W_i)$ w.r.t. X. The latter function is decreasing in X as long as $0 < \psi(X, W_i) < 1$, implying that ψ^{-1} is well defined and decreasing in such a range of capacities. Since the signal of an agent of type w satisfies $x = X + \nu\eta$, we can express her strategic belief as

$$A_w(z|\mathbf{k}^\nu, W_i) = \mathbb{P}\left(\psi^{-1}(z, W_i) \leq k^\nu(w) - \nu\eta\right) = H\left(\frac{k^\nu(w) - \psi^{-1}(z, \mathbf{k}^\nu, W_i)}{\nu}\right).$$

Differentiating w.r.t. z yields

$$\begin{aligned}\frac{\partial A_w(z|\mathbf{k}^\nu, W_i)}{\partial z} &= \frac{1}{\nu} h\left(\frac{k^\nu(w) - \psi^{-1}(z, W_i)}{\nu}\right)\left(-\frac{\partial \psi^{-1}(z, W_i)}{\partial z}\right) \\ &= \frac{h\left(\frac{k^\nu(w) - \psi^{-1}(z, W_i)}{\nu}\right)}{\frac{1}{\sum_{W_i} f(w')} \sum_{W_i} h\left(\frac{k^\nu(w') - \psi^{-1}(z, W_i)}{\nu}\right) f(w')}.\end{aligned}$$

For all $z \in (0, 1)$, we must have $h\left(\frac{k^\nu(w) - \psi^{-1}(z, W_i)}{\nu}\right) > 0$ because h is bounded away from zero in its support. Hence, the last term is positive and weakly lower than $\dfrac{\sum_{W_i} f(w')}{f(w)}$. □

[2] If $\{W_i\}_{i=1}^I$ is not kept fixed then $E(P|\mathbf{k}^\nu; k^\nu(w))$ would be discontinuous at some ν, implying a violation of the indifference condition for some $w \in \mathcal{W}^*$.

3 Data sources

Figure 1:

- Individual and corporate bankruptcies in the U.S.: U.S. courts, http://www.uscourts.gov/report-name/bankruptcy-filings.
- Corporate property executions in Italy: Italian Ministry of Justice, https://reportistica.dgstat.giustizia.it/.
- Commercial bankruptcies in Spain: Spain's Ministry of Justice, http://www6.poderjudicial.es.

4 Numerical example with three types

Here we provide a numerical example to illustrate the impulse-response in Figure 4 using our numerical example from Section 2.2.5 with three types. We additionally assume the following parameter values: $y = 1/2$, $\mu = 2/3$, $\gamma = 3/4$, $w_1 = 1$, $w_2 = 1\frac{1}{4}$, $w_3 = 1\frac{1}{3}$.

To define the initial capacity, we derive the minimal X needed to to sustain $b = 1$ in the efficient equilibrium at the enforcement stage, i.e., the equilibrium in which only type-1 agents default while types 2 and 3 repay. In such a case, since the mass of agents of type 1 is 1/3, the zero profit condition of lenders requires $(2/3)(\bar{b} - b) = (1/3)\mu b$, which gives $\bar{b} = 1\frac{1}{3}$ and which implies $\bar{w} = \bar{b}/(y + b) = 0.89$). Using equation (3), we calculate that $\theta_1 = 0.89$, $\theta_2 = 0.42$ and $\theta_3 = 0.33$.

The calculation of the strategy cutoff in Section 2.2.5 shows that types 2 and 3 will cluster on the same strategy profile at $k^*_{23} = 0.229$, representing the minimal X to sustain this equilibrium with only type-1 agents defaulting. This is also the only feasible equilibrium.

For any X lower than this level, all agents default. If all agents default, the total liquidated value of all projects is $\mu * (\frac{1}{3}w_1 + \frac{1}{3}w_2 + \frac{1}{3}w_3)b = 0.796b$, which is unfeasible to sustain. Following Figure 4, we start from $X = 0.229$ as the initial value and assume it drops to $X = 0.2$. Repeating the above calculations, we obtain from the zero profit condition that in such a case b must decline to $b = 0.905$. This implies a drop in credit of about 10%, following a drop in enforcement resources of about 13%.

References

Morris, Stephen and Hyun Song Shin. 2003. Global Games: Theory and Applications. In *Proceedings of the Eighth World Congress of the Econometric Society.*

Sákovics, József and Jakub Steiner. 2012. “Who Matters in Coordination Problems?” *American Economic Review* 102(7):3439–3461.