

Does Belief Heterogeneity Explain Asset Prices: The Case of the Longshot Bias

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Abstract

This paper studies belief heterogeneity in a benchmark competitive asset market: a market for Arrow-Debreu securities. We show that differences in agents' beliefs lead to a systematic pricing pattern, the favorite longshot bias (FLB): securities with a low payout probability are overpriced while securities with high probability payout are underpriced. We apply demand estimation techniques to betting market data, and find that the observed FLB is explained by a two-type population consisting of *canonical* traders, who hold virtually correct beliefs and are the majority type in the population (70%); and *noise* traders exhibiting significant belief dispersion. Furthermore, exploiting variation in public information across markets in our dataset, we show that our belief heterogeneity model empirically outperforms existing preference based explanations of the FLB.

JEL Classification: *C13, C51, D40, G13, L00.*

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1 Introduction

There are two important issues that have gained prominence in the study of financial markets. First, there is a growing consensus that the sheer trading volume we observe cannot be explained without resorting to information-driven trade (Cochrane, 2007), i.e., belief heterogeneity is likely a major source of gains from trade in financial markets. Second, there is also a growing body of evidence that documents “anomalies” in prices (see e.g., Keim, 2008), i.e., systematic patterns where prices do not accurately reflect the underlying fundamentals of securities.

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In this paper we show these two issues are closely related: belief heterogeneity can serve as a natural foundation for pricing patterns that depart from the predictions of standard neoclassical theory. We focus on a particular pricing ‘anomaly’, known as the favorite-longshot bias (FLB), which has been found in a number of market settings, particularly in betting markets.¹ Betting markets are important for several reasons. Chief among them is that they represent real world Arrow-Debreu security markets, which play a foundational role in the theoretical study of financial markets. In addition, unlike in other markets, we observe ex-post security returns which, combined with the availability of large datasets, allows tracing prices to fundamentals.² There is a long empirical literature that widely documents the existence of the FLB in these markets – securities with a large probability of payout (favorites) yield higher average returns than securities with a small payout probability (longshots), with observed disparities in returns as high as 200%. Because of this, the FLB is often viewed as evidence against the standard notion of agents being *risk averse*, *expected utility* maximizers with *correct beliefs* (or rational expectations) about the underlying fundamentals (Thaler and Ziemba, 1988). This has led to two main alternative approaches to explain the observed pattern of returns, both of them based on departures from standard preferences while retaining rational expectations: one approach is to use convexity of utility, i.e, risk love (Quandt, 1986), while the other approach departs from expected utility theory altogether and uses non-linear probability weighting, i.e., rank dependent utility or cumulative prospect theory preferences (Thaler and Ziemba, 1988; Jullien and Salanié, 2000; Snowberg and Wolfers, 2010).

In contrast to these preference based approaches, this paper studies the behavior of asset prices when agents have heterogeneous beliefs. In particular, we use the standard Arrow-Debreu (A-D) security market setting and explore whether belief heterogeneity, apart from rationalizing trade, can also explain the existence of the FLB. We then investigate whether our belief heterogeneity explanation can be empirically distinguished from existing preference based explanations of the FLB using data from US racetrack betting. Our main findings can be summarized as follows.

First, we theoretically show that belief heterogeneity among risk neutral traders with finite wealth naturally leads to the FLB in competitive A-D security markets: as long as the correct beliefs about the true payout probabilities lie in the support of the population belief distribution, favorites will be underpriced and longshots overpriced.

¹There is evidence of the FLB in betting and prediction markets (see e.g., Griffith (1949); Sauer (1998); Jullien and Salanié (2000); Snowberg and Wolfers (2010) for a review), option markets (where the FLB is associated to the “volatility smile”), and in some derivative markets (Tompkins et al., 2008).

²An additional key feature of betting markets is the absence of differences in liquidity or transaction cost across securities. This is in contrast with other markets where pricing patterns are closely associated to differences in liquidity and trading costs (see e.g. Sadka and Scherbina (2007) and references therein).

Second, we present a framework for estimating belief heterogeneity in A-D markets using aggregate data on prices and ex-post returns. In particular, we show that our theoretical model of belief heterogeneity can be expressed as a random utility model that we are able to non-parametrically identify from variation in prices and ex-post returns in the data. We then provide an estimation strategy by assuming that the random utility component follows a variance mixture of logit models. The variance mixture allows us to identify different types of agents in the underlying population as sub-populations with different dispersion in beliefs. Using betting market data from horse racing in the U.S., we find that a two-type population of risk-neutral traders, in which the prevalent type (about 70% of agents) holds virtually correct beliefs with minimal belief dispersion and the other type exhibits higher belief dispersion, explains the observed pricing pattern remarkably well. This provides suggestive evidence of the existence of two types of agents in these markets that trade in equilibrium - perfectly and imperfectly informed or *noise* traders.

Finally, given our empirical findings, we compare our belief heterogeneity hypothesis to existing preference based explanations for the FLB. We do so first by parametrically comparing our two-type belief model with a representative agent whose preferences follow cumulative prospect theory, which is a leading preference explanation in the literature. In particular, we use the same specification of cumulative prospect theory as the empirical analysis of Jullien and Salanié (2000), which has the same number of parameters as our two-type beliefs model and thus serves as a convenient baseline for comparison. We fail to reject the hypothesis that the predicted probabilities emerging from our model equal the true non-parametric probabilities in the data, but find a significant rejection for cumulative prospect theory. Model comparisons using Vuong non-nested hypotheses tests also supports the better explanatory power of the beliefs model.

We then aim at providing a more general contrast between our approach and existing preference explanations (which may include heterogeneity of preferences). The fundamental distinction between beliefs and preferences as a driver of choice is that beliefs should respond to information whereas preferences should not. We exploit this distinction by using a source of variation in the amount of information at races. Specifically, races that are run at the same track on the same day take two different forms, maiden and non-maiden. Because only relatively experienced horses can participate in non-maiden races and new horses (without any racing history) participate in maiden races, traders in non-maiden races have access to richer information about the value of the A-D securities associated to each horse. We find that this difference in the information structure across race types is starkly reflected in prices: maiden races exhibit a much more pronounced FLB as compared to non-maiden races. We estimate our model on each race subsample and show that it explains this pricing difference in the natural way: the same proportions of informed and

noise traders are present in both types of races, but beliefs are more dispersed in the low information races. Preference based theories that rely on homogeneous correct beliefs, on the other hand, cannot easily account for the price differences across information environments: they would require a dramatic change in agent preferences across races, which we also show cannot be reconciled by self-selection into races.

What is the intuition for why belief heterogeneity explains the FLB? Belief heterogeneity can generate a dramatic disparity in returns even when beliefs are heavily concentrated around correct beliefs because the FLB, like many pricing puzzles observed in financial markets, is driven by behavior at the tails of the underlying value distribution: disparity in expected returns becomes most apparent for securities with payout probabilities less than 1%. Accordingly, a little dispersion in beliefs suffices to induce enough demand on those longshots to generate substantial overpricing in equilibrium. For example, in our data, the average security has over twice as high an expected return as compared to extreme longshots that payout less than 1% of the time. In our estimated model, only roughly 5% of agents would prefer these longshots over the average security. But this small demand is sufficient to generate the observed overpricing.

Our findings highlight the potentially misleading inferences that can be drawn from representative agent models. Although a representative agent may rationalize the aggregate demand of a population of heterogeneous agents, her preferences need not reflect those in the population in any meaningful sense. For example, in our model, the modal behavior is well captured by the textbook rational expectations, risk neutral agent. In contrast, a representative agent exhibits non-standard preferences or beliefs to explain the same data.

Related Literature Our theoretical model is related to existing models linking belief heterogeneity to the FLB, notably the competitive model of Ali (1977) and the asymmetric information approach of Ottaviani and Sorensen (2006, 2010), but also the work of Shin (1991, 1992) and Potters and Wit (1996). The main difference between our approach and the existing literature is our focus on behavior at the tails of the value distribution, where the FLB empirically arises. In contrast, existing research typically defines the FLB for the whole range of values, e.g., by predicting returns to be strictly increasing in payout probabilities. Accordingly, we are able to substantially weaken the conditions under which the FLB arises. In addition, unlike Ottaviani and Sorensen (2006, 2010), our goal is not to provide an informational foundation to heterogeneous posterior beliefs in asset markets, but rather find general conditions on the distribution of such beliefs that lead to the FLB.

Our work is also related to a few papers showing that asset prices in a population of agents with heterogeneous beliefs need not “represent” the underlying distribution of beliefs. In particular, Manski (2006) shows that in a competitive, binary A-D security

market, the equilibrium price does not reflect the average opinion in the population. In the same setting, Ottaviani and Sorensen (2012) show that market prices systematically under-react to new information relative to the Bayesian agents in the population. Our results evoke a similar message in that we highlight the potentially misleading inferences that can be drawn from representative agent models. Nevertheless an important difference between our paper and these stems from the fact that our model links the distribution of beliefs to the underlying fundamental probabilities, which it is needed to explain the FLB. By contrast, these papers do not model the relationship between beliefs and fundamentals and focus on the aggregation of opinions rather than on the FLB.

Importantly, this paper adds a novel dimension to the empirical literature on the FLB, which, to date, has emphasized preference based explanations (Jullien and Salanié, 2000; Snowberg and Wolfers, 2010; Golec and Tamarkin, 1998). Methodologically, our estimation strategy builds on the work of Jullien and Salanié (2000), who propose a maximum likelihood framework to estimate representative agent preferences using data on prices and ex-post returns. We add heterogeneity to this framework and show that the resulting random utility model can be non-parametrically identified. A key feature of our approach is that randomness arises from belief heterogeneity, and belief based trade has not yet been explored as an empirical alternative. Relatedly, our finding that the magnitude of the FLB changes across races characterized by different information structures appears to be a new contribution to the literature on betting and financial markets.

The recent work by Chiappori et al. (2009, 2012) also add heterogeneity to the framework of Jullien and Salanié (2000) but from the perspective of preferences rather than beliefs. There are at least two important differences between their paper and our beliefs approach. First, while the discrete choice assumption on bettor behavior is without loss of generality in our model because we use risk neutral agents, it is imposed as a restriction in their setup, effectively preventing risk averse agents from constructing portfolios of bets to reduce their exposure to risk. Second, our empirical model of belief heterogeneity is isomorphic to a model of horizontally differentiated demand (with a margin of indifference between any two assets) whereas their empirical model of preference heterogeneity in risk tastes is isomorphic to a vertically differentiated demand model (with a margin of indifference only between two assets with adjacent prices). This gives rise to rather different identification and estimation strategies. Nevertheless, they also find that standard risk preferences cannot easily explain the US racetrack betting data even after accounting for preference heterogeneity, which is thus complementary to our findings.

The plan of the paper is as follows. In Section 2, we describe the market and characterize the demand of a risk neutral agent. In Section 3, we introduce belief heterogeneity and

show that it produces the FLB in equilibrium. In Section 4, we describe the dataset and illustrate the observed price pattern. We show in Section 5 how to move from the theory to an empirical framework for measuring heterogeneity. In Section 6, we describe our estimation results, while we compare our approach to the leading preference based alternatives in Section 7. Section 8 concludes.

2 The Market

We consider a competitive market for n Arrow-Debreu securities. Security i pays \$1 if outcome i takes place. Before the market opens, nature determines the state $\mathbf{p} = (p_1, \dots, p_n) \in \text{int}\Delta^{n-1}$, where $\text{int}\Delta^{n-1}$ is the interior of the $(n - 1)$ dimensional simplex and p_i is the probability that outcome i is realized. That is, Δ^{n-1} is the set of all possible probability distributions and the state $\mathbf{p} \gg 0$ is a particular probability distribution over the n possible outcomes associated to the A-D securities.

Let ρ_i denote the price of security i . We allow for the possibility of positive transaction costs represented by a fraction $\tau \geq 0$ of each dollar invested in the market that the institution keeps for its own profit. In this context, the expected gross return of investing \$1 in security $i = 1, \dots, n$ is given by

$$ER_i = (1 - \tau) \frac{p_i}{\rho_i}, \quad (1)$$

and the expected net return is $ER_i - 1$. Accordingly, securities that are underpriced relative to their true chance of yielding returns have a higher expected return as compared to securities that are overpriced.

The market consists of a population $T = [0, 1]$ with a continuum of risk neutral agents with finite endowments. Assume for the moment that there are no transaction costs ($\tau = 0$) and that the outside option yields zero net returns given agent's beliefs. The demand of agent $t \in T$ is a bundle (x_1, \dots, x_n) such that $\sum x_i \leq w_t$, where $x_i \geq 0$ is the amount invested on security i , and $w_t > 0$ is the dollar endowment of the agent.

Due to risk neutrality, the demand for security i of any agent t with well-defined posterior beliefs, denoted by $(\pi_{1t}, \dots, \pi_{nt})$, is determined by the relative comparisons of *subjective* expected returns, given by π_{it}/ρ_i , $i = 1, \dots, n$. That is, agent t will invest on the security i if it yields the highest subjective return, i.e., if $\pi_{it}/\rho_i > \pi_{jt}/\rho_j$ for all $j \neq i$.

We assume risk neutrality for two reasons. First, unlike risk aversion, it rationalizes the discrete choice of investing all the endowment in one of the securities, rather than hedging or investing only a fraction of it. This is important given that most empirical studies of the FLB take a discrete choice approach due to the fact that endowments are not observed

(see for instance Jullien and Salanié (2000), Snowberg and Wolfers, 2010). Second, people tend to be risk-neutral when stakes are low (e.g. Bombardini and Trebbi, 2010), as it is often the case for most traders in betting markets. Nonetheless, we allow for alternative risk attitudes in our estimation and find that, while introducing a risk aversion parameter improves the empirical fit, the absolute risk aversion coefficient is virtually zero in the presence of belief heterogeneity (see Section 6).

Let $r_{it} = p_i/\pi_{it}$, which reflects agent t 's relative deviation from correct beliefs about security i . The next lemma shows that the demand of agent t for security i can be characterized instead in terms of the *objective* expected returns and a vector of *indifference ratios*, given by r_{it}/r_{jt} for $j \neq i$: the agent is indifferent between security i and j whenever the ratio of expected returns ER_i/ER_j is equal to the indifference ratio r_{it}/r_{jt} .

Lemma 1. *There exists a vector $\mathbf{r}_t = (r_{1t}, \dots, r_{nt}) > 0$ with $r_{it} = p_i/\pi_{it}$ such that, for all $(\rho_1, \dots, \rho_n) \gg 0$ and all $i = 1, \dots, n$, the demand of a risk neutral agent t satisfies*

$$q_{it}(ER_1, \dots, ER_n) = \begin{cases} w_t & \frac{ER_i}{ER_j} > \frac{r_{it}}{r_{jt}} \quad \forall j \neq i \\ \{x : x \leq w_t\} & \frac{ER_i}{ER_j} \geq \frac{r_{it}}{r_{jt}} \quad \forall j, \frac{ER_i}{ER_j} = \frac{r_{it}}{r_{jt}} \text{ some } j \neq i \\ 0 & \frac{ER_i}{ER_j} < \frac{r_{it}}{r_{jt}} \text{ some } j \neq i, \end{cases}$$

and

$$\sum_i q_{it}(ER_1, \dots, ER_n) \leq w_t,$$

which holds with equality if $\frac{ER_i}{ER_j} \neq \frac{r_{it}}{r_{jt}}$ for some i, j .

Proof. In the *Appendix*. □

Notice that an agent with correct beliefs has indifference ratios equal to one for all i, j . Likewise, if an agent has an indifference ratio $r_{1t}/r_{2t} = 2$, then asset 1 would have to pay twice as high objective expected returns as asset 2 for the agent to be indifferent between the two (thus the agent either seriously overestimates the probability of asset two paying off, or underestimates asset 1).

There are two advantages of working with objective returns and indifference ratios rather than dealing directly with prices and subjective beliefs. First, the fact that indifference ratios reflect relative differences with respect to the true underlying payout probabilities allows us to define (idiosyncratic) belief heterogeneity in terms of deviations from correct beliefs (see *Assumption 1* below). Second, the indifference conditions in *Lemma 1* readily lead to the random utility model we use later in our empirical analysis.

Finally, let s_i be security i 's market share, i.e., the amount invested on security i relative to the total (finite) amount invested in the market. Market clearing implies that prices equal market shares, as long as agents, when indifferent between investing or not, choose to invest either all their endowment or nothing.³ It is worth noting that, by definition, prices are equal to shares in parimutuel markets, as are the markets comprising our dataset.

3 Belief Heterogeneity and Prices

In a standard representative agent model, traders are homogeneous in terms of beliefs and preferences. In particular, they hold correct posterior beliefs about the state of the world. We call such (risk-neutral) agents *canonical traders*, and their individual demand is characterized by $\mathbf{r}_t = (1, \dots, 1)$. In this context, the only trading equilibrium outcome is $ER_i = ER_j$ for all i, j , since otherwise the market does not clear.⁴ That is, there do not exist any gains from trade in this economy and agents are indifferent between investing in any of the securities or staying out of the market and thus such equilibrium is not robust to the introduction of transaction costs.

Now consider introducing gains from trade in this baseline model by letting agents exhibit differences in posterior beliefs, i.e., heterogeneity in indifference ratios. Let the mass function of \mathbf{r}_t be given by the conditional probability measure $P[\cdot \mid \mathbf{p}, \theta]$, which depends both on the state of the world p and possibly on other characteristics of the market $\theta \in \Theta$. We only require that there is always a positive mass to “both” sides of the canonical trader –this avoids the no-trade trap discussed above. To capture this condition, given $z \in \mathbb{R}_+^{n-1}$, let

$$L_i[z \mid \mathbf{p}, \theta] = P \left[\left(\frac{r_{it}}{r_{1t}}, \dots, \frac{r_{it}}{r_{i-1t}}, \frac{r_{it}}{r_{i+1t}}, \dots, \frac{r_{it}}{r_{nt}} \right) \ll z \mid \mathbf{p}, \theta \right], \text{ for } i = 2, \dots, n-1,$$

and define L_1 and L_n in a similar fashion. $L_i[z \mid \mathbf{p}, \theta]$ represents the mass of agents with indifference ratios associated to security i lower than z , i.e., $r_i/r_j < z_j$ for $j \neq i$. For instance, $L_i[(1, \dots, 1) \mid \mathbf{p}, \theta]$ is the mass of agents that strictly prefer i over any other

³Market clearing means the supply of dollars equals the demand of dollars in each of the possible n outcomes. This happens if and only if

$$\frac{q_i}{\rho_i} = (q_1 + \dots + q_n) \iff \rho_i = s_i = \frac{q_i}{q_1 + \dots + q_n} \quad (\forall i),$$

where q_i represents the aggregate investment in dollars on security i . If a mass of agents is indifferent between investing in some securities and staying out of the market, and they invest only a fraction of their endowment then the supply of dollars may be higher than the total investment in the market, in which case some agents are short-selling some of the securities.

⁴If $ER_i > \max_{j \neq i} ER_j$ for some i then all traders invest in security j , leading to $\rho_i = 1$ and $\rho_j = 0$ for all $j \neq i$, implying $ER_j > ER_i$, a contradiction.

security when expected returns are equal across securities. In this context, we assume that there is enough heterogeneity so that demand for security i is bounded away from 0 and 1 when $\tau = 0$ and $ER_i = 1$ for all $i = 1, \dots, n$. That is, there is minimal “liquidity” at fair prices regardless of the market characteristics (\mathbf{p}, θ) .

Assumption 1. [Idiosyncratic belief heterogeneity] $L_i[(1, \dots, 1) | \cdot]$ is bounded away from zero for $i = 1, \dots, n$.

It is worth emphasizing that this assumption allows for the support of L_i to converge to a single point as $p_i \rightarrow 0$, as long as all the mass is not concentrated at $(1, \dots, 1)$ when $p_i > 0$. For instance, L_i could have full support in $[1 - \varepsilon, 1 + \varepsilon]^{n-1}$ with $\varepsilon \rightarrow 0$ as $p_i \rightarrow 0$. In other words, it allows for subjective beliefs to be absolutely continuous with respect to true probabilities, and it does not implicitly require the distribution of subjective beliefs to exhibit “fat tails.” In addition, as we show in the Appendix, *Assumption 1* can be weakened when endowments and beliefs are independently distributed, by letting the distribution of beliefs exhibit “vanishing tails,” and our results would still go through.

The next result shows that heterogeneity induces the FLB, even in the presence of transaction costs. In particular, any security i with a sufficiently high value (p_i) is underpriced in equilibrium, while the remaining (low value) securities are overpriced on average.

Theorem 1. [FLB] *If Assumption 1 holds, there exists $\bar{\tau} > 0$ such that for all $\tau < \bar{\tau}$ a necessary consequence of equilibrium is that there exists $\bar{q} < 1$ such that, for all $i = 1, \dots, n$, if $p_i > \bar{q}$ then security i is underpriced while securities $j \neq i$ are overpriced on average.*

Proof. In the Appendix □

Why does the FLB arise when agents are heterogeneous? Three key features in our model help explain it. On the institutional side, the underlying value of an Arrow-Debreu security is bounded, since it is given by its payout probability. On the agent side, we assume that agents have finite endowments and are risk neutral. Hence, they focus on relative comparisons of (subjective) expected returns across securities, choosing to invest their endowment on the highest return security. Finally, we assume that belief heterogeneity is idiosyncratic, i.e. represented by some dispersion of subjective expected returns around the true underlying expected returns. In this context, risk neutrality ensures that demand for each security does not vanish when true expected returns are equal across securities, regardless of payout probabilities. As a consequence, the presence of this minimal demand pushes the price of a security above its payout probability whenever the latter is very small, causing the security to be overpriced –the underpricing of securities with a high payout probability directly follows by noting that prices in the market add up to one.

We provide some intuition by focusing on the case of two securities, unit endowments ($w_t = 1$ for all $t \in T = [0, 1]$) and zero transaction costs. Note that, in this context, if the returns across the securities are (ER_1, ER_2) , then the share s_1 of investment in security 1 is bounded above and below by

$$L_1[ER_1/ER_2 \mid \mathbf{p}, \theta] \leq s_1 \leq 1 - L_2[ER_2/ER_1 \mid \mathbf{p}, \theta].$$

These bounds are tight because agents with $r_{1t}/r_{2t} = ER_1/ER_2$ are indifferent towards any investment (x_1, x_2) with $0 \leq x_1 + x_2 \leq 1$.

In this market idiosyncratic heterogeneity implies the existence of a positive lower bound of demand for security 1 when $ER_1 = ER_2$, given by $\underline{q} = \inf\{L_1[1 \mid \mathbf{p}, \theta] : (\mathbf{p}, \theta) \in \text{int}\Delta \times \Theta\}$. But then, when p_1 is sufficiently low, this lower bound \underline{q} on demand prevents the share of security i and thus its price to fall below p_1 , leading to overpricing of security 1 ($ER_1 < 1$) and underpricing of security 2 ($ER_2 > 1$ since $p_2 = 1 - p_1 < 1 - s_1 = s_2$). More formally, let $p_1 < \underline{q}$ and suppose the theorem does not hold, i.e., $ER_1 \geq ER_2$. Then, by the above bounds we must have $s_1 \geq L_1[1 \mid \mathbf{p}, \theta] \geq \underline{q} > p_1$. But this is a contradiction since $s_1 > p_1$ implies that $ER_1 < ER_2$.⁵ A similar intuition applies when p_1 is sufficiently high, given the upper bound $1 - L_2$.

It is worthwhile to note the generality of our condition. Idiosyncratic heterogeneity only requires that there is some dispersion of beliefs around the correct beliefs, i.e. that there are minimal gains from trade in each security at fair prices.

To illustrate the potential of belief heterogeneity to generate the FLB consider the following example. Suppose there are two assets in the market with probability of paying out a dollar being .9% and 99.1%, respectively, and there are 99 risk neutral agents with accurate beliefs and only one risk neutral agent who believes that the probability of payout on the longshot is anything better than 1%. If all agents have unit endowments, then the trading equilibrium involves this single agent investing in the longshot while all the remaining agents invest in the favorite. In this case, the A-D price is .01 for asset 1 and .99 for asset 2. Thus a very small departure in beliefs of one percent of the population creates a sizable disparity in returns: the expected return on the longshot in the heterogeneous agent economy is -10 percent whereas the favorite has a positive expected return.

Theorem 1 states that the FLB arises for extreme probabilities. However, it would be interesting to know under what conditions a global FLB arises, i.e., that securities with higher payout probabilities exhibit higher equilibrium returns. As the next result shows, one way to obtain the global FLB is to impose a symmetry condition on the distribution

⁵Notice that all that we need is $L_1[1 \mid \mathbf{p}, \theta] > \alpha p_1$ for some $\alpha > 1$ and all p_i small enough. Hence, we can replace the uniform bound on L_1 with this “vanishing bound,” as long as $s_1 \geq L_1$ holds, which is the case when endowments and beliefs are independent. See Appendix A.1 for details.

of beliefs. Specifically, to require that, whenever asset i equilibrium returns are at least as high as asset j 's, the mass of agents believing that asset i yields the highest returns is at least as large as those believing the best asset is j .

Assumption 2. $P[\cdot | \mathbf{p}, \theta]$ is atomless for all (\mathbf{p}, θ) . Moreover, if $ER_i \geq ER_j$ then

$$L_i \left[\frac{ER_i}{ER_1}, \dots, \frac{ER_i}{ER_n} \mid \mathbf{p}, \theta \right] \geq L_j \left[\frac{ER_j}{ER_1}, \dots, \frac{ER_j}{ER_n} \mid \mathbf{p}, \theta \right]$$

for all $i, j = 2, \dots, n$, with a similar condition for $i, j \in \{1, n\}$.

Theorem 2. [Global FLB] *If Assumption 2 holds, there exists $\bar{\tau} > 0$ such that, for all $\tau < \bar{\tau}$, $p_i < p_j$ implies $ER_i < ER_j$ in equilibrium.*

Proof. We only proof the case of $\tau = 0$. The proof for $\tau < \bar{\tau}$ follows the same argument used in the proof of *Theorem 1* and is therefore omitted.

First, notice that P being atomless implies that if equilibrium expected returns are given by ER_i for all i then market shares satisfy $s_i = L_i \left[\frac{ER_i}{ER_1}, \dots, \frac{ER_i}{ER_n} \mid \mathbf{p}, \theta \right]$. Now consider, by way of contradiction, that $p_i < p_j$ but $ER_i \geq ER_j$. Then it must be the case that $s_i < s_j$ since $ER_i = p_i/s_i \geq p_j/s_j = ER_j$. But this implies that

$$L_i \left[\frac{ER_i}{ER_1}, \dots, \frac{ER_i}{ER_n} \mid \mathbf{p}, \theta \right] < L_j \left[\frac{ER_j}{ER_1}, \dots, \frac{ER_j}{ER_n} \mid \mathbf{p}, \theta \right],$$

a violation of Assumption 2. □

This result is a generalization of Ali (1977) to the case of $n \geq 2$ horses. He shows that the global FLB arises in a market with two securities whenever $Pr(r_{1t}/r_{2t} \leq 1 \mid \mathbf{p}, \theta) = .5$ for any (\mathbf{p}, θ) , i.e., the canonical belief is the median belief in the population. To see why Theorem 2 generalizes this result notice that, when $n = 2$, by Assumption 2 we must have that $L_1[1 \mid \mathbf{p}, \theta] = L_2[1 \mid \mathbf{p}, \theta] = 1 - L_1[1 \mid \mathbf{p}, \theta]$, i.e., $L_1[1 \mid \mathbf{p}, \theta] = 1/2$. Using a similar argument it is easy to show that Assumption 2 implies Assumption 1: at fair prices we must have that $L_i[(1, \dots, 1) \mid \mathbf{p}, \theta] = L_j[(1, \dots, 1) \mid \mathbf{p}, \theta]$ for all i, j . But since P is atomless then $L_i[(1, \dots, 1) \mid \mathbf{p}, \theta] = 1/n$ for all i and all (\mathbf{p}, θ) .

4 Data

Betting markets, and racetrack betting in particular, are textbook illustrations of Arrow-Debreu securities markets. We focus our attention on the “win odds” market, which is the market for bets on which horse will win. This is considered the most competitive market

at the racetrack, given it has the most liquid pool of money. Not surprisingly, it has been the subject of the most empirical attention.

Prices at the racetrack are quoted in terms of the odds R_i on horse or security $i = 1, \dots, n$, which are defined as the *net* return per dollar bet on security i in the event i wins the race –the *gross* return is given by $R_i + 1$. In North American racetracks, the odds are determined through a “parimutuel” system of wagering in which the losers pay the winners. This system ensures that there are no differences in liquidity across securities and is equivalent to the market clearing condition in an Arrow-Debreu securities market where shares equal prices. Accordingly, market odds satisfy

$$R_i = \frac{1 - \tau}{s_i} - 1 \quad i = 1, \dots, n. \quad (2)$$

Given a vector of observed odds (R_1, \dots, R_n) , we can invert (2) to recover the underlying A-D prices $\rho_i = s_i$ for $i = 1, \dots, n$.

The FLB is in fact a widely documented empirical pattern of returns across bets at the racetrack. To illustrate it, consider a large data set consisting of horse starts, i.e, a sample of horses that competed in some race. Each observation i corresponds to a pair $(win_i, R_i + 1)$, where $R_i + 1$ is the gross return that A-D security i pays conditional on horse i winning and win_i is an indicator variable for whether horse i won the race or not. Define A_i as the ex-post gross return on security i , i.e.,

$$A_i = \begin{cases} R_i + 1 & \text{if } win_i = 1 \\ 0 & \text{if } win_i = 0, \end{cases}$$

and thus the regression

$$E[A_i | \log s_i] = E[(R_i + 1) win_i | \log s_i] = f(\log s_i) \quad (3)$$

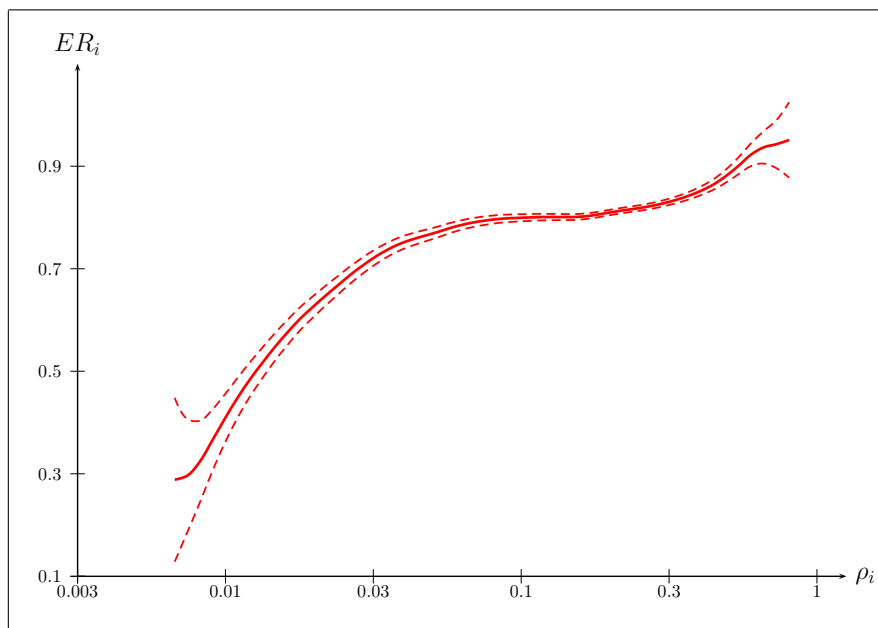
measures the expected gross return among securities that have the same log price ($\log s_i$). Using a sample of 176,652 races that were collected from North American tracks over 2003-2006,⁶ consisting of 1,456,512 horse starts, we estimate f by non-parametrically regressing A_i on $\log s_i$ using a locally linear kernel weighted regression.⁷ *Figure 1* shows the estimated returns –along with their 95% confidence interval, which clearly exhibit the FLB: low value securities have lower expected returns, where the effect is particularly pronounced when comparing the extremes. The returns in *Figure 1* are net of the track take. Thus, since

⁶The data were collected through the efforts of members at the website paceadvantage.com. This data set is one of the largest to have been assembled to study betting behavior.

⁷We use the Epanechnikov kernel and a rule of thumb bandwidth estimator.

the average track take is about 0.19, these returns imply that, roughly, horses with prices below 0.2 are overpriced and those with prices higher than 0.2 are underpriced.

Figure 1: Favorite Longshot Bias

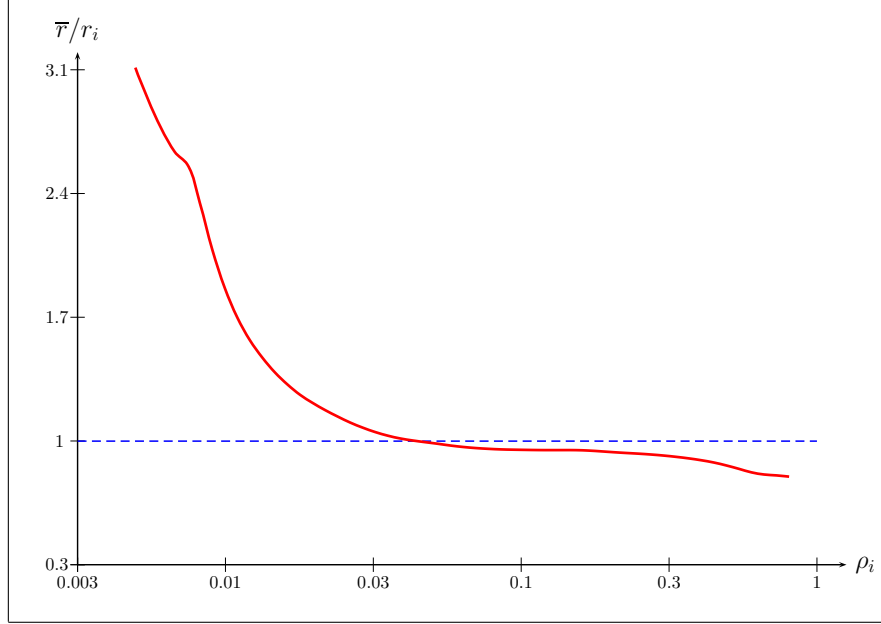


To understand the argument implicitly put forth by Thaler and Ziemba (1988) as to why the FLB as found in the data is so puzzling, recall Lemma 1 and consider the indifference ratios of a representative agent that clears the market in equilibrium. In equilibrium, the agent must be indifferent across securities to sustain positive asset demand. The expected gross return to an average bet in the data is $ER_{avg} = .76$. We can thus use *Figure 1* to recover the representative agent's indifference ratio ($\frac{\bar{r}}{r_\rho}$), which is the ratio associated to the 'average' security (representing a random bet) and a security with price ρ . This ratio represents the amount that the average security must pay in expected returns relative to a security with price ρ in order for the agent to be indifferent, and we can back out this relationship from the data because the representative agent is indifferent across all securities for the market to clear. As *Figure 2* shows, the representative agent overvalues securities with prices .01 or less by over 50 percent, and securities with prices .005 or less by approximately 100 percent, compared to the average bet.

5 Estimating Heterogeneity

We now show how to estimate a model of an A-D economy where agents have heterogeneous beliefs. The goal is to estimate the pattern of belief heterogeneity that is consistent with

Figure 2: Representative Agent Indifference Ratios



both the betting data and equilibrium in the model. That is, we seek to understand whether a heterogeneous beliefs model can explain the observed FLB and, furthermore, what the estimated heterogeneity looks like in contrast with the above representative agent.

In order to illustrate our empirical strategy we first describe the general maximum likelihood (ML) approach for estimating preferences from aggregate betting data, which was first introduced by Jullien and Salanié (2000) (which we shall abbreviate as JS) in the context of a representative agent framework. We then show how to extend their framework to a population of heterogeneous agents. Let us recall that the dataset consists of a sample of K markets or races, which are assumed to be independent of each other. Each market $k = 1, \dots, K$ is defined by the number of securities n^k , a vector of odds $\mathbf{R}^k = (R_1^k, \dots, R_{n^k}^k)$, and the identity of the winner i_{win}^k . The data thus identify the empirical relationship between prices and fundamentals, i.e., for any number of securities n with odds (R_1, \dots, R_n) we can identify the underlying probabilities of winning $\mathbf{p}(R_1, \dots, R_n)$: intuitively, $p_i(R_1, \dots, R_n)$ is identified by the fraction of times horse i wins in the subset of races characterized by (R_1, \dots, R_n) . Formally, this can be identified by the non-parametric regression of the indicator variable that a horse i wins on the vector of odds in the race.

A key insight in JS is to recognize that an equilibrium model of the betting market, described by parameters $\theta \in \Theta$, implies a relationship between prices and fundamentals, denoted $\mathbf{p}(R_1, \dots, R_n; \theta)$. The empirical strategy implicitly underlying JS is to find the parameters $\theta \in \Theta$ whose prediction about the equilibrium relationship between prices and

fundamentals best matches the actual empirical relationship. This strategy is naturally implemented using maximum likelihood, and can be described by two key steps:

1. Given a choice of model parameters $\theta \in \Theta$, for each market k , find the vector of *unobserved* payout probabilities $\mathbf{p}^k(\theta) = (p_1^k(\theta), \dots, p_{n^k}^k(\theta))$ that is consistent with *observed* odds \mathbf{R}^k under equilibrium. That is, we solve

$$\mathbf{p}^k(\theta) = \phi(\mathbf{R}^k, \theta), \quad k = 1, \dots, K, \quad (4)$$

where ϕ is a model-specific mapping from odds to probabilities.

2. Estimate θ by maximizing the log-likelihood

$$LL(\theta) = \sum_{k=1}^K \log p_{win}^k(\theta). \quad (5)$$

This ML estimator consistently estimates the value of $\theta_0 \in \Theta$ that minimizes the Kullback-Leibler distance between the model $\mathbf{p}(R_1, \dots, R_n; \theta_0)$ and the data $\mathbf{p}(R_1, \dots, R_n)$.⁸

JS showed that, in a representative agent framework of market equilibrium, there exists a unique equilibrium mapping between prices and fundamentals that is required for Step 1. In particular, the equilibrium mapping is determined by the set of indifference conditions

$$U(p_1^k, R_1^k; \theta) = U(p_2^k, R_2^k; \theta) = \dots = U(p_J^k, R_J^k; \theta), \quad (6)$$

where $U(p_i^k, R_i^k; \theta)$ is the payoff to the agent from investing in a security with payout probability p_i^k at odds R_i^k . That is, \mathbf{p}^k is solved to make the agent indifferent about investing across securities at the observed odds. JS formally show that, if U is continuous and respects first order stochastic dominance, the mapping ϕ from odds to probabilities implied by (6) is well-defined. Using these conditions, preference parameters θ , such as risk aversion or cumulative prospect theory (CPT) coefficients, can be estimated. Using this approach, JS compared different representative agent preferences, and found evidence in favor of CPT over risk loving as the preferred explanation of the FLB.

A heterogeneous agents model, however, cannot be approached using this representative agent framework: the set of agents who are indifferent across all securities in a heterogeneous population typically has measure zero, as is the case when the distribution of beliefs is continuous. Thus we can no longer use the equilibrium conditions (6) to solve for the relationship between prices and probabilities $\mathbf{p}^k(\theta) = \phi(\mathbf{R}^k, \theta)$ in a heterogeneous

⁸Standard errors follow from standard likelihood theory, which JS give explicitly for this context in Section 4.c of their paper.

agents model (where the parameter $\theta \in \Theta$ determines the pattern of heterogeneity in the underlying betting population). Instead, the additional information that we will use to solve for the equilibrium relationship between prices and fundamentals are the market shares of the securities $\mathbf{s}^k = (s_1^k, \dots, s_n^k)$, which are derived from observed odds using (2). Note that market shares are not needed in the representative agent approach. However a heterogeneous agents model predicts market shares in a natural way.

In particular, if U_t denotes the payoff function of agent t and we assume that she invests her endowment on a single security,⁹ market shares are given by the aggregation of individual investments:

$$s_i^k = \frac{1}{\int_T w_t dt} \int_T w_t \mathbf{1} \left[U_t(p_i^k, R_i^k) > U_t(p_j^k, R_j^k) \forall j \neq i \right] dt, \quad i = 1, \dots, n^k, \quad (7)$$

where $\mathbf{1}[\cdot]$ is the indicator function. Thus, our goal is to characterize U_t in terms of our model of belief heterogeneity and show that the system of equations (7) can be uniquely inverted to recover \mathbf{p}^k . In order to be consistent with our theoretical model and to ease exposition, we focus here on the case of a population of risk neutral agents. Nonetheless, we describe in the Appendix the general identification strategy when the trader population exhibits heterogeneity in both risk preferences and beliefs.

To characterize (7) under our model of risk neutrality and heterogeneous beliefs, consider an generic race with n horses. Let us return to *Lemma 1*, which allows us to represent an agent t 's preferences in terms of the relative deviations from correct beliefs $\mathbf{r}_t = (r_{1t}, \dots, r_{nt}) > 0$ for any agent $t \in T$. The model implies that agent t invests his/her endowment in security $i = 1, \dots, n$ if

$$\frac{ER_i}{ER_j} \geq \frac{r_{ti}}{r_{tj}} \quad \forall j \neq i.$$

Assuming a continuous distribution of $\mathbf{r}_t = (r_{1t}, \dots, r_{nt}) > 0$ in the population (so that indifference between two securities has measure zero), risk neutrality gives rise to a discrete choice model for asset demand consistent with (7). The fact that the discrete choice behavior is without loss of generality in our belief heterogeneity model is a desirable feature of our approach and not shared with the existing preference based literature that must impose discrete choice behavior as an additional assumption.

Because choices are invariant to a monotonic transformation of preferences, we can

⁹While our theoretical analysis allows agents to abstain from the market when $\tau > 0$, here we assume that participation in the market is exogenously given and focus only on the decision of which asset to invest. This is primarily driven by the data: we do not have information on the fraction of consumers who do not participate in the market. Rather, our data only gives us demand conditional on participation. This approach is also shared by the literature on estimation of preferences with betting data.

express agent t 's utility for betting on asset i in the market as

$$U_t(p_i, R_i) = u_{ti} = \delta_i + \nu_{it}. \quad (8)$$

where $\delta_i = \log ER_i - \log ER_1$ and $\nu_{it} = \log r_{1t} - \log r_{it}$. Notice the utility of horse 1 has been normalized so that $u_{1t} = 0$. Given this utility specification, it is straightforward to check that $U_t(p_i, R_i) > U_t(p_j, R_j)$ is equivalent to $\frac{ER_i}{ER_j} \geq \frac{r_{ti}}{r_{tj}}$. Importantly, $\nu_t = (\nu_{1t}, \dots, \nu_{nt})$ fully characterizes an agent's beliefs. To see why, observe that $\log p_i - \log p_1 + \nu_{it} = \log(\pi_{ti}) - \log(\pi_{t1})$ for each $i = 2, \dots, n$, where recall π_{it} is agent t 's subjective belief that horse i wins. Combined with the constraint that $\sum_{i=1}^n \pi_{it} = 1$, we can uniquely recover the vector of subjective beliefs $\pi_t = (\pi_{1t}, \dots, \pi_{nt})$ from ν_t (we have n equations in n unknowns). Thus, we can represent an agent's beliefs in terms of the vector of additive errors ν_t .

Let $P(\cdot; \theta)$ be the distribution of ν_t that captures the belief distribution in the population. P is indexed by an unknown vector of parameters $\theta \in \Theta$ that is the object of estimation. Assuming that P is continuous and that the distribution of endowments is independent of P , by *Lemma 1* we can express (7) as

$$s_i = \int_{\nu_t \in \mathbb{R}^n} \mathbf{1}[\delta_i + \nu_{it} > \delta_j + \nu_{jt} \forall j \neq i] dP(\nu_{1t}, \dots, \nu_{nt}; \theta) \quad i = 1, \dots, n, \quad (9)$$

It turns out that this system of equations (9) is isomorphic to a *horizontally* differentiated product market in which each agent chooses among n products, with δ_i representing the “mean utility” of product i —the mean utility of product 1 is normalized to be zero—and $\nu_t = (\nu_{1t}, \dots, \nu_{nt})$ is a vector of random utility terms that is heterogeneous in the population—the random utility term ν_{1t} of product 1 also being normalized to be zero. Inverting systems of equations defined by (9) is central to the demand estimation framework put forth by Berry et al. (1995) (aka BLP), i.e., solving for mean utilities given the underlying distribution of preferences over a set of differentiated products using the observed prices and market shares over these products. This connection provides us an immediately useful result: so long as P is continuous, then Berry (1994) shows that there exists a vector of mean utilities $(\delta_1, \dots, \delta_n)$ that solves (9). Furthermore, the more recent Berry et al. (2012) shows that the solution is unique.

We can now exploit these results to carry out the empirical strategy of JS in a heterogeneous agents context. Constructing the likelihood for given choice of parameters θ can be described by the following steps.

- (a) For any market k , we numerically find the unique underlying vector of mean utilities $\delta^k = (\delta_1^k, \dots, \delta_n^k) \in \mathbb{R}^n$ that solves the system of equations (9).

(b) Given $\delta^k = (\delta_1^k, \dots, \delta_{n^k}^k)$, we can recover the underlying probability distribution over states of nature \mathbf{p}^k by using the following facts

- (i) $\delta_i^k = \log ER_i^k - \log ER_1^k$,
- (ii) The expected gross return is $ER_i^k = (1 - \tau) \frac{p_i^k}{s_i^k}$,
- (iii) Probabilities $(p_1^k, \dots, p_{n^k}^k)$ over the n^k possible states of nature sum up to one.

6 Results

The only aspect of the estimation that remains to be discussed is how to parameterize $P(\nu_t; \theta)$. To discipline the parametric model we employ, we impose that belief heterogeneity is *idiosyncratic*. That is, we restrict P to satisfy $E[\nu_{it}] = 0$ for all $i = 1, \dots, n$, and

$$E[\nu_{it} \mid \nu_{-i,t}] = 0 \tag{10}$$

for any $\nu_{-i,t} \in \mathbb{R}^{n-1}$. We interpret (10) as an important restriction for the random utility terms ν_{it} to represent belief heterogeneity rather than preference heterogeneity. Under preference heterogeneity, if agent t has a large a preference shock for one horse j (i.e., a high ν_{jt}) it could mean that she also has a high ν_{it} for another horse i with similar characteristics as j in the race. Thus, even if $E(\nu_{it}) = 0$, preference heterogeneity would not necessarily satisfy (10). Accordingly, condition (10) prevents agents from exhibiting systematic tastes for horses, which could possibly create an artificial demand for longshots.

We introduce now a rich yet tractable way to model belief heterogeneity that respects idiosyncratic heterogeneity. In particular, we let the random utility terms ν_{it} be distributed according to a *variance mixture* of logistic errors. That is,

$$P(\nu_{2t}, \dots, \nu_{nt}; \theta) = \int \prod_{i=2}^n F(\nu_{it} \mid \sigma) dG(\sigma),$$

where $G(\sigma)$ is the mixing distribution. This mixture, which appears new to the discrete choice literature, retains the key properties of idiosyncratic heterogeneity (10).¹⁰

We show in the Appendix (section D) that the distribution G can be non-parametrically identified from the data. Here we discuss how G is estimated. Solving for the inner integral in (9) analytically (see e.g., Train (2003)) yields the well known mixed logit demand

$$s_i = \int_{\sigma \in \mathbb{R}_+} \frac{\exp\left(\frac{1}{\sigma} \delta_i\right)}{\sum_{j=1}^n \exp\left(\frac{1}{\sigma} \delta_j\right)} dG(\sigma) \quad i = 1, \dots, n, \tag{11}$$

¹⁰Conditional on σ , $F(\nu_{2t}, \dots, \nu_{nt} \mid \sigma)$ represents i.i.d. logit errors with common variance σ and thus satisfies idiosyncratic heterogeneity. Then, by integrating out over σ , idiosyncratic heterogeneity is retained.

where the mixing takes place over the distribution of the variances σ . Although this system (11) bears a resemblance to the mixed logit demand that was originally estimated by Berry et al. (1995), there is a critical difference. In contrast to the usual mean mixture –which violates (10), our heterogeneity is governed by a variance mixture, thereby introducing a random coefficient $\frac{1}{\sigma}$ on the mean utility term δ_i . This random coefficient does not affect the existence and uniqueness of a solution to the system, but it does affect our ability to compute it because the contraction mapping proposed in Berry et al. (1995) is no longer valid (the mean utilities cannot interact with random coefficients in their setup). Instead, we minimize the sum of squared errors between observed shares and predicted shares using a quasi-newton procedure with zero mean utilities as initial values.

A one component mixture, i.e., $G(\sigma)$ with only one point in its support, corresponds to the standard logit. Adding components gives rise to a finite mixture. We view this as a natural way to capture heterogeneity in the population as the different components can be interpreted as trader “types.” A finite mixture with J components corresponds to a parameter vector $\theta = (\sigma_1, \dots, \sigma_J; P_1, \dots, P_J)$, where σ_j for $j = 1, \dots, J$ are the support points of the finite mixture and P_j is the probability mass of component j .¹¹ The parameter estimates for a one, two and three component mixture are presented in *Table 1*.

The first thing to note is that the two-component specification provides a much better fit than the one-component specification. To see this fact, we illustrate in *Figure 3* the pattern of (net) expected returns implied by the one-component and the two-component specifications, respectively, which are obtained by regressing the predicted ER_i^k on $\log s_i^k$ for all securities i and markets k in the data. While the one component model does a poor job fitting the expected returns of longshots, the two-component model fits the observed pattern remarkably well. This improvement is further confirmed by the likelihood ratio test, which strongly rejects the one-component in favor of the two-component model.

Table 1 also shows that adding a third component does not significantly improve the log likelihood. In addition, it roughly reproduces the two-component by splitting the predominant component ($P_1 = 0.716$) into two low variance types (1 and 3) with the second component still exhibiting substantial dispersion. Consequently, in what follows, we focus on the two-type model.

In order to check our assumption of risk neutrality, we also estimate a two-component model in which agents have (homogeneous) CARA preferences following the general approach laid out in the Appendix. The last column of *Table 1* shows the model estimates. The estimated CARA coefficient (denoted by γ) yields a tiny degree of risk aversion ($\gamma = 0.003$) albeit not significantly different from zero. In addition, the remaining parame-

¹¹Any distribution over a compact support can be approximated by a multinomial distribution. See e.g., *Theorem 10.15* in Aliprantis and Border (2006).

Table 1: Parameter Estimates

Parameter	One Type Estimate (std. error)	Two Types Estimate (std. error)	Three Types Estimate (std. error)	Two Types-CARA Estimate (std. error)
σ_1	0.067 (0.0035)	0.028 (0.0033)	0.014 (0.0058)	0.034 (0.0074)
σ_2	-	0.503 (0.0588)	0.690 (0.1757)	0.514 (0.0862)
σ_3	-	-	0.075 (0.0306)	-
P_1	1	0.716	0.461 (0.1237)	0.706
P_2	-	0.284 (0.0184)	0.219 (0.0390)	0.294 (0.020)
P_3	-	-	0.320	-
CARA(γ)	-	-	-	0.003 (0.0031)
Log-likelihood	-307,391.5	-307,291.8	-307,291.3	-307,288.0
LR Test ^a (p-value)	199.4 (<0.0001)	-	1.0 (0.6065)	7.6 (0.0058)

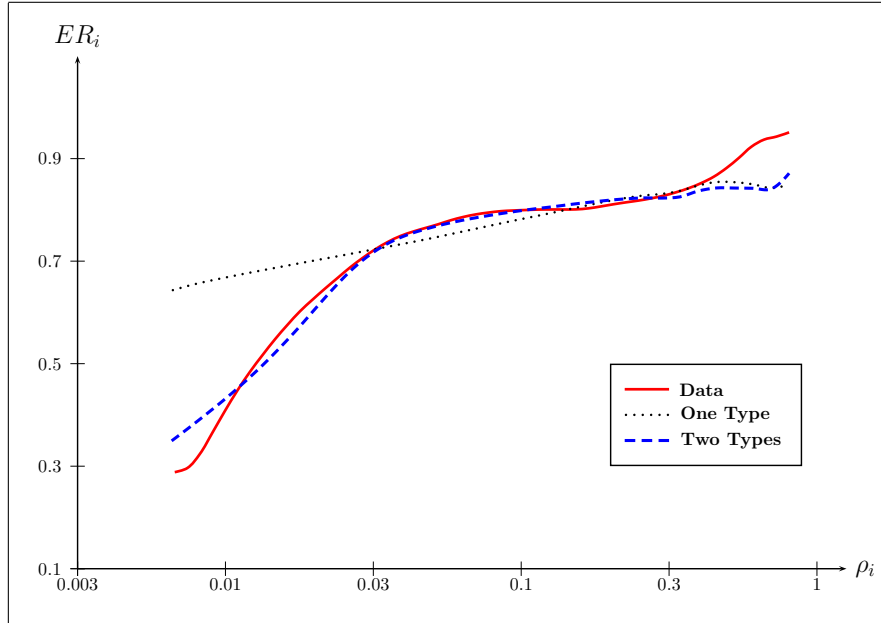
^aThe LR test statistic compares the model against the two-type specification and is given by $2 \ln LL_a - 2 \ln LL_0$, where LL_0 and LL_a represent the log-likelihood of the null and the alternative model, respectively (where the alternative model nests the null).

ters are virtually identical to the two-component risk neutral version of the model, implying that the economic implications of these two models are virtually identical. Nonetheless, allowing for CARA preferences yields a statistically significant likelihood gain.

6.1 Economic Interpretation: Informed and Noise Traders

The two-component mixture suggests a bimodality in the underlying distribution of agents in the market. In the context of financial markets, there is a large literature that theoretically introduces two types of agents: *arbitrageurs* who are relatively well informed, and traders exhibiting dispersed beliefs due to poor information or imperfect belief updating. This two-type classification is commonplace in the market microstructure literature (see e.g. Glosten and Milgrom (1985), Kyle, 1985), which separates traders into “noise” traders and insider speculators, in the ‘limits of arbitrage’ literature (De Long et al., 1990; Stein, 2009), and in research related to mispricing in speculative markets (Shin, 1991, 1992;

Figure 3: Predicted and Actual Returns



Serrano-Padial, 2012). Although a pure noise trader in the literature is an agent that invests randomly, we use the term noise trader in the spirit of De Long et al. (1990), i.e., to refer to traders exhibiting belief dispersion. Ottaviani and Sorensen (2006) interpret these types in betting markets as *late* (arbitrageurs) and *early* traders (recreational gamblers).

If we view the components of the mixture as representing different “types” of agents, then type σ_1 represents informed traders, which is also the modal type in the population (72%), while type σ_2 is a noise trader, since it exhibits a much higher dispersion in subjective valuations. This is, to our knowledge, the first paper empirically suggesting the existence of informed and noise traders using only aggregate data.¹²

To quantify the dispersion of type- σ_2 agents, consider the choice between investing in a security at random, yielding average returns in expectation, and an extreme longshot. For example, in our dataset, the average (gross) return is 0.76, whereas it is about 0.35 for securities in the bottom percentile of the price distribution. The probability that a type- σ_2 agent prefers a security with average returns to a security i yielding 0.35 is given by

$$Pr(0.76/0.35 > \bar{r}_t/r_{it}) = Pr\left(\bar{\nu}_t - \nu_{it} < \frac{(\log 0.76 - \log 0.35)}{0.503}\right) \approx 0.82,$$

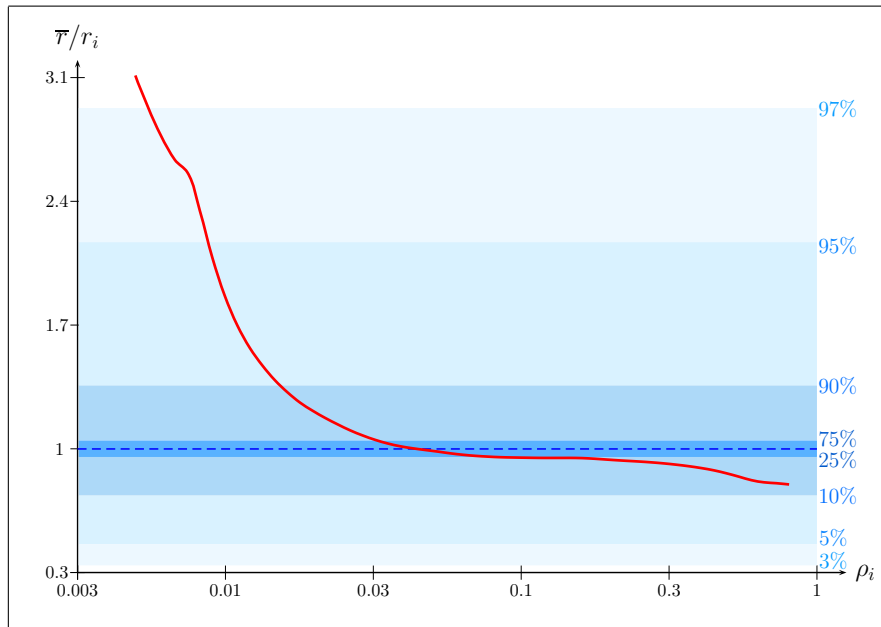
¹²There are several empirical studies that estimate the presence of noise traders using individual trading data from experimental markets (Forsythe et al., 1992; Cipriani and Guarino, 2005; Cowgill et al., 2009), or transaction data from financial markets (Easley et al., 1997).

where the last equality follows from the fact that $\bar{\nu}_t - \nu_{it}$ is distributed standard logistic.

Furthermore, since type σ_2 only represent 28% of the population, and type σ_1 prefer the favorite approximately virtually 100 percent of the time, the probability of the random investment being preferred to the longshot in the population as a whole is approximately $0.82 \times 0.28 + 0.72 \approx 0.95$. This is in stark contrast with a representative agent, who must be indifferent between the two bets in order to explain the data.

We further compare the representative agent and traders in our estimated model by looking at the distribution of indifference ratios $\frac{\bar{\tau}}{r_p}$ in the two-component population. *Figure 4* shows different quantiles of the distribution, which we compare alongside the implied indifference ratios of a representative agent. Agents in the interquartile range, i.e., the inner 50% of the agents in the population, exhibit very little dispersion from the canonical agent, i.e., the agent with indifference ratios equal to 1. That is, the modal behavior in the heterogeneous population is closely captured by a risk neutral agent with correct beliefs.¹³ This is driven by the fact that the first component has a very low variance. Beliefs become more dispersed for agents outside the interquartile range, reflecting the fact that the variance of the second component is much higher ($\sigma_2 = 0.503$ versus $\sigma_1 = 0.028$). It is only at the far tails of our estimated distribution that we approach the extreme beliefs of the representative agent with respect to longshots.

Figure 4: Individual vs. Market Behavior



¹³The interquartile range is a standard measure of dispersion and modal behavior. Manski (2004) also uses it to document dispersion in beliefs.

Figure 4 also provides intuition for why the two-type specification performs better than the one-type. Consistent with the numerical example in Section 3, to generate the observed disparity in returns between longshots and the average bet, we need only a small fraction of traders investing in the longshot. This is because the empirical winning probabilities of longshots are very small. Accordingly, most of the agents belonging to the first component exhibit beliefs that induce them to invest on the favorites, while agents in the second component spread their investment across securities, with a minority of them going for the extreme longshots. In contrast, the one-type specification faces a trade-off: either it exhibits low belief dispersion (as it is the case) thus fitting well the returns on securities with moderate and high payout probabilities but does not generate enough demand for the longshots, or it exhibits higher dispersion, hence fitting poorly the returns on favorites.

7 Preferences vs. Belief Heterogeneity

We now contrast our belief heterogeneity approach with preference based explanations of the FLB in two different ways. First, we formally compare the empirical fit of our estimated model to the standard preference explanation in the literature, probability overweighting, which JS and Snowberg and Wolfers (2010) have shown to explain the data better than risk loving. Second, we exploit the presence of exogenous variation in the information structure across races to non-parametrically test the predictions of our belief heterogeneity hypotheses against the predictions from a general class of preference based theories.

We engage in formal model comparisons by using the representative agent model of CPT preferences as was employed by JS and assess its performance, relative to our model, in terms of explaining the variation in the data. The CPT model we estimate is the preferred specification in JS: a three-parameter model consisting of a single CARA utility function with risk coefficient γ for both monetary gains and losses, and two separate probability weighting functions: one for gains and the other for losses –this specification is richer than the one-parameter rank-dependent utility of Snowberg and Wolfers (2010). As in JS, we assume that the representative agent has correct beliefs about \mathbf{p} . The value of investing w on security i given odds R_i is given by

$$U(p_i, R_i) = G(p_i)u(wR_i, \gamma) + H(1 - p_i)u(-w, \gamma),$$

where $u(x, \gamma)$ is a CARA utility, $G(p) = p^\alpha$ is the weighting function for gains and $H(p) = p^\beta$ is the weighting function for losses.¹⁴ We estimate the model by maximum likelihood

¹⁴Observe that a single initial wealth level does not have to be assumed here because the CPT model considers only the utility of gains and losses relative to existing wealth.

Table 2: CPT Estimates

	Full Sample	JS	non-Maiden	Maiden
Parameter	Estimate (std. error)	Estimate (std. error)	Estimate (std. error)	Estimate (std. error)
CARA (γ)	-0.032 (0.0006)	-0.072 (0.021)	-0.020 (0.0029)	-0.032 (0.0041)
Gains (α)	1.22 (0.0307)	1.162 (0.143)	1.12 (0.0307)	1.20 (0.0420)
Losses (β)	0.28 (0.0006)	0.318 (0.272)	0.55 (0.0898)	0.25 (0.0769)
Observations	176,466	4,037	87,394	29,003
Log-likelihood	-307,301.5	-7,365.3	-156,101.0	-50,508.8
Vuong Test (p-value)	34.92 (<0.0001)			

using indifference conditions (6) –see JS for details.

Our parameter estimates along with the CPT estimates from JS are presented in the first two columns of *Table 2*. As can be seen, both estimates are quite close, but because of the much larger size of our sample our estimates have a much higher precision. We find that preferences exhibit slight risk loving ($\gamma < 0$), a slightly convex weighting of gains ($\alpha > 1$) and a highly concave weighting of losses ($\beta \ll 1$). Our estimates reinforce the major empirical finding from JS: under CPT, the key primitive driving the overpricing of longshots is the overweighting of loss probabilities (particularly for small probabilities).

7.1 Testing the Models

Observe that the likelihood of our model exceeds that of the CPT model. This suggests that our beliefs approach exceeds the explanatory power of the existing CPT explanation. We can formalize this comparison in light of our identification strategy. Let $\phi(\mathbf{R})$ denote the *true* mapping between prices and fundamentals in the data, i.e., $\mathbf{p} = \phi(\mathbf{R})$, and denote $\phi^m(\mathbf{R}; \theta)$ the mapping between prices and fundamentals predicted by model $m \in \{\text{Belief heterogeneity, CPT}\}$ for $\theta \in \Theta^m$. If model m is properly specified, then

$$\phi(\mathbf{R}) = \phi^m(\mathbf{R}; \theta^m) \tag{12}$$

for some ‘true value’ of parameters $\theta^M \in \Theta^M$. A natural question is whether (12) is actually testable, i.e., whether we can test that model m is consistent with the underlying data generating process. The likelihood framework naturally allows for such a specification

test, which was first proposed by White (1982). This specification test is derived from the information matrix equality, which is a fundamental theorem in likelihood theory (see e.g., Cameron and Trivedi, 2005). In our context, the information equality states that, if (12) holds, the following must be satisfied:

$$E \left[\frac{\partial^2}{\partial \theta \partial \theta'} \log \phi_{i_{win}}^m(\mathbf{R}; \theta^m) \right] = -E \left[\frac{\partial}{\partial \theta} \log \phi_{i_{win}}^m(\mathbf{R}; \theta^m) \frac{\partial}{\partial \theta} \log \phi_{i_{win}}^m(\mathbf{R}; \theta^m)' \right], \quad (13)$$

where the expectation is taken with respect to the true data generating process over winners and odds (i_{win}, \mathbf{R}) in the data. White's test uses the sample analogues of each side of the equality to construct a quadratic form that should be sufficiently close to zero if (12) is true. We use the form of the test given by White (1987). Specifically, for each race k in the data let $l_k(\hat{\theta}^m) = \log \left(\phi_{i_{win}}^m(\mathbf{R}^k; \hat{\theta}^m) \right)$, i.e., the log likelihood value associated with race k at the estimated parameters $\hat{\theta}^m$. The test statistic is constructed as

$$T^m = \left\{ K^{-1/2} \sum_{k=1}^K \hat{q}_k \right\}' \left\{ \frac{1}{K} \sum_{k=1}^K \hat{q}_k \hat{q}_k' \right\}^{-1} \left\{ K^{-1/2} \sum_{k=1}^K \hat{q}_k \right\},$$

where

$$\hat{q}_k = \text{vech} \left\{ \frac{\partial^2 l_k}{\partial \theta \partial \theta'}(\hat{\theta}^m) + \frac{\partial l_k}{\partial \theta} \frac{\partial l_k}{\partial \theta'}(\hat{\theta}^m) \right\}.^{15}$$

If (12) holds, T^m is asymptotically distributed χ^2 with $J(J+1)/2$ degrees of freedom, where J is the length of the parameter vector $\theta^m \in \Theta^m$. Hence large values of T^m indicate a significant difference between the model's predictions and the data.

The results of the test are rather stark: it fails to reject (12) for the belief heterogeneity model at standard significance levels ($T = 8.54$ whereas the 10% critical value is 10.64). Hence, we cannot reject the null hypothesis that the BH model equals the true data generating process. In contrast, $T^{CPT} = 1,139.08$, which clearly rejects the null that the CPT model describes the true data generating process. Hence, although the log-likelihood values of the two models are somewhat close, as far as the information equality test there is a big difference in terms of their ability to accurately describe the true data generating process. We don't interpret this test as literally telling us that our model is entirely properly specified. However, we do interpret it as saying that it captures the subtle variation in the data well enough such that it cannot be distinguished from the true data generating process with nearly 200,000 races and only 3 parameters! We reach similar conclusions if instead we implement the non-nested model selection tests proposed by Vuong (1989) between the BH and CPT models (see bottom of *Table 2*).

¹⁵vech(A) denotes the half-vectorization of symmetric matrix A .

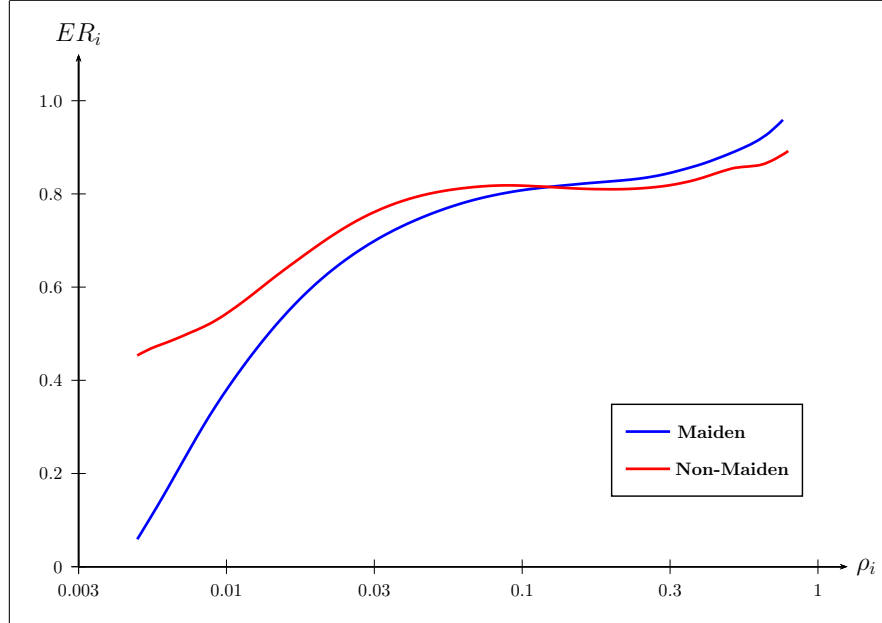
7.2 Information Differences and the FLB

We have shown that our estimated two-type model of belief heterogeneity better explains the FLB as compared to a representative agent CPT model. However, these model comparison tests have a few drawbacks. First, we have only offered one parametric alternative (the CPT model estimated by JS), which is far from exhausting the space of possible preference alternatives, including possibly heterogeneous preference models such as the one studied by Chiappori et al. (2012). Moreover, although we derived our random utility (11) on the basis of belief heterogeneity, and our estimates have a natural interpretation in terms of informed and noise traders, such heterogeneity could also have a preference interpretation.

In light of this, how can we distinguish between our belief based explanation and preference based models? The key difference is the following: in a standard preference based approach, an individual has preferences for binary gambles (p, R) and thus each individual's preferred asset in a race depends deterministically on the menu of gambles $\{(p_i, R_i)\}$ in the race. In our belief based model this is no longer true: the true underlying menu of gambles is insufficient to characterize an individual's asset demand because an individual will have idiosyncratic deviations in beliefs. These belief shocks will vary for the same individual across both horses and races. Thus, for the same menu of gambles $\{(p_i, R_i)\}$ in two different races, an agent will have different preference ranking over horses $i = 1, \dots, n$ in the two races depending on the belief shocks she receives, whereas in preference based models the individual will have exactly the same preferences ranking over horses as long as the underlying menu of gambles is invariant across races.

This distinction has a very important testable implication: beliefs change with information whereas preferences for gambles do not. Thus the equilibrium prices predicted by our beliefs based model will be different for two races with the same fundamentals but different public information whereas the predictions from a preference based model will be the same. This immediately suggests a non-parametric approach to testing between our beliefs model and preference alternatives - an ideal experiment would be to use exogenous variation in the amount of information about participating horses and empirically study whether this impacts pricing in the market. Unfortunately, our data does not contain any horse specific information. However, we exploit the fact that horse races on the same day at the same track come in two different forms: *maiden* and *non-maiden*. Horses in a maiden race are those that have yet to win a single race, and thus by definition any new horses are entered into maiden races whereas horses must have a racing history to participate in a non-maiden race. Thus, on average, there is much less handicapping information about horses in maiden races as compared to non-maiden (Camerer, 1998; Mitchell, 1989). Accordingly, for a given vector of winning probabilities \mathbf{p} and the same trader population,

Figure 5: The FLB in Maiden and non-Maiden Races



the lower uncertainty surrounding non-maiden races should lead to less belief dispersion and, in turn, to a less pronounced FLB, since traders can condition their beliefs on richer information – a fact we discuss in more detail below. In contrast, a purely preference based model would yield identical predictions across race types.

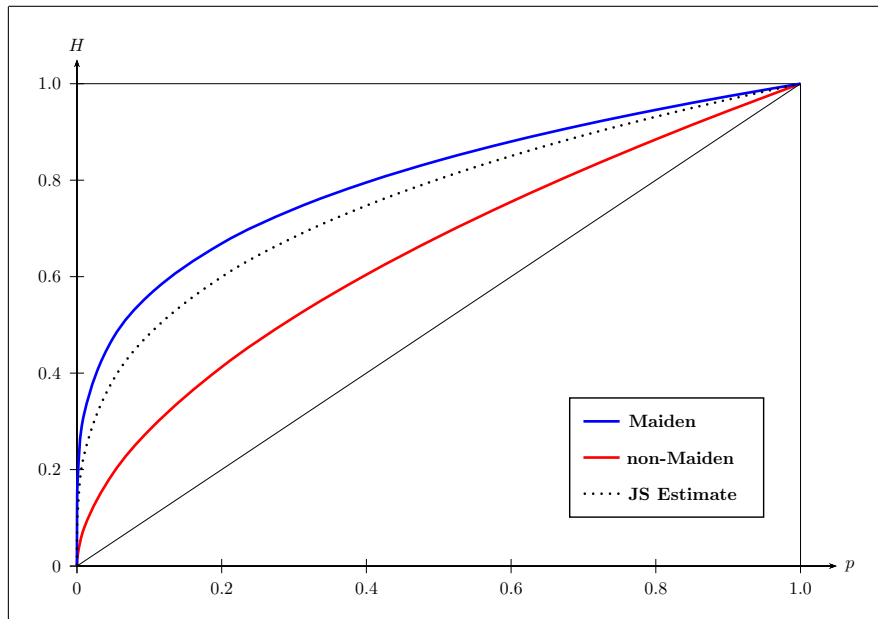
To compare these predictions, we control for other differences that might exist between maiden and non-maiden races by restricting our samples to *claiming* races, in which horses can be purchased before the races and exhibit horses with similar price tags (thus trying to ensure a level playing field). This is the most frequent type of races in the US (over 54% of all races). Furthermore, maiden and non-maiden claiming races take place typically on the same day at the same track,¹⁶ and have similar track takes.¹⁷ Thus, arguably the trader population should be similar across maiden and non-maiden claiming races.

Figure 5 shows the relationship between prices and returns for each race type. As can be seen, there are considerable price differences. Specifically, the magnitude of the FLB is much more pronounced in maiden compared to non-maiden races: while expected returns in maiden races go from \$0.05 for extreme longshots to \$0.96 for heavy favorites, returns in maiden races are much more compressed, ranging from \$0.45 to \$0.89, with the reduction in mispricing being especially prominent for longshots.

¹⁶In our dataset, 99.5% of maiden races were run in a day where non-maiden races were also run.

¹⁷The average track take for maiden claiming races is 0.190 with std.dev. 0.04. For non-maiden claiming races the average take is 0.192 and the std.dev. is 0.04.

Figure 6: CPT Weighting Function for Losses



This result hints at a basic inconsistency of a preference based view of the FLB: it can only explain pricing differences across types of races through variation in preferences. We illustrate the magnitude of this inconsistency by estimating the above CPT model on the separate subsamples. The estimates are given in the last two columns of *Table 2*, which show significant differences across samples, in particular on the probability weighting of losses (significant at the 1% level): while both the full sample and maiden parameters are very similar, β is more than twice as high in non-maiden races, resulting in a much less concave weighting function (see *Figure 6*). These large swings in preferences across maiden and non-maiden races are difficult to rationalize and, given the stark differences in prices across races, such swings will also be present in any alternative preference based model.¹⁸

7.2.1 Self-Selection into Races

We now show that this preference instability observed across race types is not easily explained by the self selection of agents into races based on heterogeneous risk preferences. The hypothesis is that, in addition to making an optimal choice of which horse to bet *within* a race, agents with different risk preferences are also selecting *across* races in a way that accounts for the pricing differences between maiden and non-maiden races.

This self-selection hypothesis has a clear testable implication. If agents are self-selecting

¹⁸Following Chiappori et al. (2009), we also estimated a model with heterogeneous risk attitudes and found that while most agents are risk lovers in maiden races, they become risk averse in non-maiden races.

based upon *well-behaved* risk preferences, then we should not observe any significant arbitrage opportunities across race types that would allow a bettor to receive higher odds at higher win probabilities. This is because all standard risk preferences, including expected utility and CPT, satisfy first order stochastic dominance, meaning that agents prefer bets that exhibit both lower prices (higher odds) and lower risk (higher winning probability). Thus, if such opportunities exist, bettors should exploit them by switching bets across race types until prices adjust to the point where this dominance is eliminated. We will call the existence of such an arbitrage a “profitable switch.”

The picture in *Figure 5* already suggests a large presence of profitable switches in the data: longshots in non-maiden races pay higher average returns than similar longshots in non-maiden races. This pricing asymmetry can be exploited to construct a betting strategy that switches bets from maiden to non-maiden races and earns both higher odds at higher win probabilities. In order to construct such a strategy, we use our two-type specification to estimate the underlying winning probabilities \mathbf{p}^k for each race k in the data –recall we fail to reject it is the true data generating process, so true probabilities should be very close to the ones predicted by the model. We then check for every horse in a maiden race whether there is a horse in a “nearby” non-maiden race that dominates or is dominated by the maiden horse (where “nearby” means a race that takes on the same day at the same track, and “dominance” means an asset pays higher odds with a higher winning probability than another). Thus, whenever a maiden horse is dominated by a non-maiden one in a nearby race, our model suggest a profitable switch from maiden to non-maiden that takes advantage of the fact that similar longshots are more favorably priced in non-maiden races.

We find that 91.6% of maiden races have at least one violation of dominance, and 81.5% include a profitable switch to a non-maiden race according to our model, that is, a horse i with price $\rho_j^{k'} < \rho_i^k$ and probability $p_j \geq p_i$ for some horse j in a non-maiden race k' run in the same day at the same track. If we restrict attention to adjacent maiden and non-maiden races, i.e., maiden races immediately preceded or followed by a non-maiden race in the same track, the percentage of violations is still quite high: 63.5%, with 49.5% of them including a profitable switch to a non-maiden race. By comparison, the frequencies of dominance violations across maiden races are only 31.1% for same day races and 22.3% for adjacent races. Consequently, our model detects pervasive violations of dominance across race types, compared to violations within type, that would appear strongly at odds with selection on preferences being the cause of the pricing differences between races.

The dominance violations flagged by our model translate into a significant arbitrage opportunity. The second row of *Table 3* shows that using the flags to switch from maiden to non-maiden in the same day at the same track leads to a 7.3% increase in the probability of winning and a 5% increase for adjacent switches. Because every switch involves a cheaper

Table 3: Alternative Betting Strategy

Variable	Same Day		Adjacent	
	Estimate ^a (Robust s.e.)	Relative Change	Estimate (Robust s.e.)	Relative Change
Δ Ex post Returns	0.0935** (0.0231)	29.19%	0.0715* (0.0429)	23.85%
Δ Winner ^b	0.0042** (0.0012)	7.33%	0.0030* (0.0023)	4.98%
Δ Log Price	-0.0468** (<0.0001)		-0.0612** (<0.0001)	
Observations	65252		20375	

^aStd. errors are clustered by race. ** means p -value < 0.01 , * means p -value < 0.1 .

^bOne sided test. Null Hypothesis: Δ Winner < 0 .

bet in a non-maiden race, traders would pay lower prices at no cost to the overall probability of winning. This generates a large effect on the average returns to betting: as the first row in *Table 3* illustrates, ex post returns increase by 29.2% (23.8% for adjacent races).

These estimates show that selection into races based upon risk preferences alone cannot explain the preference instability across maiden and non-maiden races since any such selection should exploit the substantial arbitrage opportunities present across race types, and thus reduce the observed pricing differences rather than create them. The only selection effect involving preference that could explain the price variation is one based upon *both* risk attitudes and non-pecuniary preferences. In particular, the price differences between maiden and non-maiden races requires individuals betting in maiden races to be less risk averse. This margin of selection may seem natural. However, to avoid the "no arbitrage" argument above, there must also be an unexplained preference shock for betting in maiden races. One difficulty with this explanation is that, if there were non-pecuniary preference shocks, e.g., a thrill to betting on one's favorite horse, these are likely to be more pronounced for horses with a longer racing history rather than new horses.

7.2.2 Can Belief Heterogeneity Explain the Difference?

We now show that heterogeneous beliefs provides a natural explanation for the pricing disparity between maiden and non-maiden races. Since there is less public information in maiden races we should expect more belief dispersion in maiden races, particularly among noise traders, leading to a more pronounced FLB. Nonetheless, our beliefs model does still impose a restriction we can test: the change in the FLB across race types should be driven

by a change in the belief dispersion of each type and *not* by a change in the proportion of trader types. If our model instead were to empirically require a higher prevalence of noise traders in the population to explain the change in the pattern of returns, it would exhibit the same parameter instability that the preference based theories suffer from. We formulate this claim as two testable hypotheses.

Hypothesis 1. *The fraction of types is the same across maiden and non-maiden races.*

Hypothesis 2. *Belief dispersion of each type is higher in maiden races compared to non-maiden races.*

The basic insight behind *Hypothesis 2* is that, if agents are Bayesian, their beliefs should get closer as the amount of (public) information about the underlying state of the world accumulates over time. For instance, in the context of horse racing, if agents observe the performance of horses over time, they would eventually agree on the (true) winning probabilities.¹⁹ Accordingly, if beliefs get closer as more information becomes available, we should also expect a smaller FLB in non-maiden races. We show in the Appendix that this is the case in a model where the same A-D security market is repeated over time. Specifically, we show that, for given \mathbf{p} , the FLB gets mitigated when agents can observe the history of ex-post returns and agree on the *direction* of the belief updating.

Table 4 presents the model estimates for the two samples, which confirm *Hypothesis 1* and *2*. The fraction of type one is very stable around 70-73% –we fail to reject *Hypothesis 1* at any standard significance level. In contrast, belief dispersion estimates are significantly smaller (at the 1% level) in non-maiden races: σ_1 is virtually zero and σ_2 is about half its value in maiden races. Thus, remarkably, our heterogeneous beliefs approach explains the change in prices across information structures in the theoretically predicted way.

Overall, these results give credence to the idea that belief heterogeneity is a major driver behind the FLB because, unlike preference theories, it is able to naturally explain large changes in pricing patterns across races with different information environments. It is important to stress that the disparity between maiden and non-maiden prices does not rule out alternative representative agent models, such as those in which preferences exhibit some form of ambiguity aversion, since one can regard bets on a maiden horse as more ambiguous than bets on a non-maiden horse. For instance, agent preferences could follow CPT for uncertainty in which the weighting function is replaced by a non-additive capacity (see Tversky and Kahneman, 1992). Nonetheless, any such model would still need

¹⁹Belief convergence is formally treated in the literature on merging of opinions (Blackwell and Dubins, 1962; Kalai and Lehrer, 1994; Lehrer and Smorodinsky, 1996; Gossner and Tomala, 2008), and models of learning in competitive markets (Townsend, 1978; Feldman, 1987; Easley and Blume, 1982; Vives, 1993). Although the mentioned research on merging concerns the evolution of beliefs in the long run, Sandroni and Smorodinsky (1999) show that the speed of convergence can be quite fast.

Table 4: BH Estimates for Claim and non-Claim Races

Parameter	non-Maiden Estimate (std. error)	Maiden Estimate (std. error)	Full Sample Estimate (std. error)
σ_1	0.0001 (<0.0001)	0.0366 (0.0079)	0.028 (0.0033)
σ_2	0.340 (0.0180)	0.6599 (0.1403)	0.503 (0.0588)
P_1	0.700	0.727	0.716
P_2	0.300 (0.0104)	0.273 (0.0321)	0.284 (0.0184)
Observations	87,394	29,003	176,466
Log-likelihood	-156,090.0	-50,505.8	-307,291.8

to explain the differences across race categories by means of a change in beliefs, regardless of whether those beliefs are represented by additive probabilities or not. In addition, it would exhibit the same weaknesses as a model in which the representative agent has expected utility preferences but her beliefs are chosen ad hoc to rationalize the data.

8 Conclusion

We have shown in this paper that allowing for belief heterogeneity in asset markets can reconcile model predictions with observed aggregate patterns without compromising the validity of standard behavioral assumptions such as weak risk aversion and expected utility. We have used Arrow-Debreu security markets as a general setting in which to illustrate these points. We have also showed that a belief based model of trade in financial markets outperforms existing preference based explanations of the FLB on several dimensions. We think heterogeneous beliefs have the potential to play an important role in empirical work in other institutional settings, such as insurance and credit markets, and we hope this paper encourages future research in this area.

Appendix

A Omitted Proofs

Proof of Lemma 1. Consider the maximization problem of agent t with endowment $w_t > 0$ and beliefs $(\pi_{1t}, \dots, \pi_{nt}) \gg 0$. Given the state of the world (p_1, \dots, p_n) and market prices $(\rho_1, \dots, \rho_n) \gg 0$, the agent solves

$$\begin{aligned} \max_{(x_1, \dots, x_n) \in \mathbb{R}_+^n} \quad & \sum_{i=1}^n x_i \left(\frac{\pi_{it}}{\rho_i} - 1 \right) + w_t \\ \text{s.t.} \quad & \sum_{i=1}^n x_i \leq w_t. \end{aligned}$$

The ratio π_{it}/ρ_i represents the *subjective* (expected) returns of security i , i.e. the return given agent's beliefs. If there is a security i such that

$$\frac{\pi_{it}/\rho_i}{\pi_{jt}/\rho_j} > 1, \quad \forall j \neq i, \quad (14)$$

then the solution to the agent's problem implies investing all the endowment w_t in security i . This is because (i) security i yields the strictly highest subjective returns among all securities; and (ii) security i 's subjective returns are strictly greater than one. Property (ii) comes from the fact that, since $\sum_i \pi_i = 1$ and $\sum_i s_i = 1$, we must have $\max_h \frac{\pi_{ht}}{s_h} > 1$ whenever $\frac{\pi_{ht}}{s_h} \neq \frac{\pi_{kt}}{s_k}$ for some h, k . This also means that, if the latter is true, the agent will invest all her endowment in the securities yielding $\max_h \pi_{ht}/s_h$, being indifferent about how much to invest on each of them. Finally, if $\frac{\pi_{ht}}{s_h} = \frac{\pi_{kt}}{s_k}$ for all h, k , then subjective returns are all equal to one, making the agent indifferent between investing any amount in $[0, w_t]$. To finish this case, notice that, by letting $r_{it} = p_i/\pi_{it}$, expression (14) becomes

$$\frac{ER_i}{ER_j} > \frac{r_{it}}{r_{jt}}, \quad \forall j \neq i.$$

□

Proof of Theorem 1. We consider first the case of a market with zero transaction costs $\tau = 0$. Let $q_i(ER_1, \dots, ER_n; \mathbf{p}, \theta)$ represent the aggregate investment on asset i and denote $\mathbf{r}_{it} = \left(\frac{r_{it}}{r_{1t}}, \dots, \frac{r_{it}}{r_{i-1t}}, \frac{r_{it}}{r_{i+1t}}, \dots, \frac{r_{it}}{r_{nt}} \right)$. Given *Lemma 1*, q_i at fair prices satisfies, for all (\mathbf{p}, θ) ,

$$\int_T w_t \mathbf{1}[\mathbf{r}_{it} < (1, \dots, 1)] dt \leq q_i(1, \dots, 1; \mathbf{p}, \theta) \leq \int_T w_t \mathbf{1}[\mathbf{r}_{it} \leq (1, \dots, 1)] dt.$$

By *Assumption 1*, the lower bound on q_i in this expression is bounded away from zero while the upper bound is bounded below $\int_T w_t dt$. In addition, aggregate investment is bounded by the aggregate endowment in the market:

$$\sum_j q_j(ER_1, \dots, ER_n; \mathbf{p}, \theta) \leq \int_T w_t dt.$$

This implies that, the market share of security i at fair prices, which is given by

$$s_i(1, \dots, 1; \mathbf{p}, \theta) = \frac{q_i(1, \dots, 1; \mathbf{p}, \theta)}{\sum_j q_j(1, \dots, 1; \mathbf{p}, \theta)},$$

must be bounded above zero and below one. Define

$$\underline{s} := \min_i \inf \{s_i(1, \dots, 1; \mathbf{p}, \theta) : (\mathbf{p}, \theta) \in (0, 1) \times \Theta\} > 0.$$

To prove part (i) of the theorem, let $p_i > \bar{q} := 1 - \underline{s}$ and suppose the theorem does not hold, i.e., $p_i \leq \rho_i = s_i$. Then, since $\sum_{j \neq i} s_j = 1 - s_i$ and $\sum_{j \neq i} p_j = 1 - p_i$ there must be at least a security h with $ER_h \geq 1$. Pick the one with the highest expected returns. For this security we must have $\frac{ER_h}{ER_j} \geq 1$ for all j , and thus it must be that $s_h \geq \underline{s}$. But then, since $s_i \geq p_i > 1 - \underline{s}$ by assumption, we have that $s_h \leq \sum_{j \neq i} s_j < \underline{s}$, a contradiction. Obviously if security i is strictly underpriced, we must have that the remaining securities are overpriced on average: $\sum_{j \neq i} \rho_j > \sum_{j \neq i} p_j$.

Now consider the introduction of transaction costs. If, before introducing them, trader t was indifferent between investing on any security i or not investing,²⁰ we must have

$$\frac{ER_i}{ER_j} = \frac{r_{it}}{r_{jt}} \quad \forall j \neq i. \quad (15)$$

In this context, let $ER_i = \frac{p}{s_i^*}$, where s_i^* is the market share associated with zero transaction costs. After introducing positive transaction costs, for the agent to be indifferent between investing in security i and not investing in the market, expected returns need to satisfy

$$ER_i = (1 - \tau) \frac{p_i}{s_i'} = \frac{p}{s_i^*},$$

where s_i' is the new share for i , implying that $s_i' < s_i^*$. But this means that (i) the agent does not longer invest in the market at the old prices ($\rho_i = s_i^*$ for all i), and (ii) if she is indifferent between investing in i or not at the new prices, she has strict incentives not to

²⁰Recall from the proof of *Lemma 1* that for an agent to be indifferent between investing or not she must also be indifferent between investing among any two securities.

invest in some of the other securities. This is because $s'_i < s_i^*$ implies $s'_j > s_j^*$ for some security j , given that market shares add up to one. Hence, the above indifference condition translates into

$$\frac{ER_i}{ER_j} \geq \frac{r_{it}}{r_{jt}} \quad \forall j \neq i, \quad (16)$$

with strict inequality for some j .

Summing up, introducing transaction costs can potentially reduce the demand for any given asset. If this reduction is big enough the equilibrium market share for some security may not be bounded above zero, also implying that the other securities' shares may not be bounded below one, so that the FLB may not hold. Thus, in order to show that the FLB holds in equilibrium we need to show that this does not happen for low enough τ .

The first thing to note is that, since $L_i[z|\mathbf{p}, \theta] = Pr[\mathbf{r}_{it} \ll z|\mathbf{p}, \theta]$ and $L_i[1|\mathbf{p}, \theta]$ is bounded away from zero, we can always find $z = (z_1, \dots, z_n) \ll (1, \dots, 1)$ close enough to $(1, \dots, 1)$ such that $L_i[z|\mathbf{p}, \theta]$ is also bounded above from zero for all i .

Second, notice that the smaller the track take τ , the closer the market share s'_i gets to s_i^* with $s'_i \rightarrow s_i^*$ as $\tau \rightarrow 0$ for all i , and thus the closer the indifference condition (16) gets to condition (15). Hence, if we fix $ER_i/ER_j = 1$ for all i, j , given $z \ll (1, \dots, 1)$ we can always find a low enough $\bar{\tau} > 0$ such that, for all $\tau < \bar{\tau}$, the marginal traders indifferent between investing in i or not have indifference ratios satisfying $z < \mathbf{r}_{it} < 1$.

Therefore, combining these two facts we can show that market shares when $ER_i/ER_j = 1$ for all i, j are bounded away from zero for all $\tau < \bar{\tau}$ given some small $\bar{\tau} > 0$ and apply the same reasoning as in the case of $\tau = 0$ to show that the FLB must hold in equilibrium. \square

A.1 Weakening Assumption 1

As we mention in Section 3, when endowments and beliefs are independent, the FLB could obtain even when the mass of agents strictly preferring security i over the alternatives dwindles to zero as $p_i \rightarrow 0$. The next assumption and theorem formalize this intuition.

Assumption 3. *There exist $\underline{p} > 0$ and $\alpha > 1$ such that $L_i[(1, \dots, 1) | \mathbf{p}, \theta] > \alpha p_i$ for all $p_i < \underline{p}$ and all $i = 1, \dots, n$.*

Theorem 3. *If Assumption 3 holds and endowments are independent of P , there exists $\bar{\tau} > 0$ such that for all $\tau < \bar{\tau}$ a necessary consequence of equilibrium is that there exists $\bar{q} < 1$ such that, for all $i = 1, \dots, n$, if $p_i > \bar{q}$ then security i is underpriced while securities $j \neq i$ are overpriced on average.*

Proof. The same argument as the one used in the proof of *Theorem 1* follows through by noticing that the orthogonality of endowments and beliefs implies that s_i is bounded below

by L_i . To see why, focus on the case of $\tau = 0$ and notice that

$$\begin{aligned} s_i &\geq \frac{1}{\int_T w_t dt} \int_T w_t \mathbf{1}\{r_{it}/r_{jt} < ER_i/ER_j \forall j \neq i\} dt \\ &= \frac{1}{\int_T w_t dt} \left(\int_T w_t dt \right) L_i[(\dots, ER_i/ER_j, \dots) \mid \mathbf{p}, \theta], \end{aligned}$$

where the last inequality follows from the independence of endowments and indifference ratios. But then, the above assumption guarantees that $s_i \geq L_i[(1, \dots, 1) \mid \mathbf{p}, \theta] > p_i$ at fair prices for all \mathbf{p} with $p_i < \underline{p}$. Given this, the same argument by contradiction used in the proof of *Theorem 1* immediately applies. \square

B Public Information and the FLB

In this section we illustrate how the FLB is mitigated by the arrival of information, which is consistent with the above findings regarding maiden and non-maiden races.

Consider the following scenario. There is going to be a sequence of two A-D security markets involving the same securities –the same result readily generalizes to having more than two markets. True payout probabilities \mathbf{p} are independent and the same across markets. We assume that the first time the market is run, agents have heterogeneous posterior beliefs. We also assume that the distribution of those beliefs is continuous with full support in $int\Delta^{n-1}$ and that $\tau = 0$ and that, in the absence of public information, agents' inferences and equilibrium prices in both markets would be the same. Markets are identical, *except* in the amount of public information available: before trading in the second market, all agents observe the realized outcome in the first market, i.e., which security paid positive returns. Let \mathcal{H}_i denote the event that security i pays out in the first market. Given agent t 's beliefs, let $\mathcal{L}_{ij}^t(\mathcal{H}_i)$ denote the likelihood ratio associated to outcome \mathcal{H}_i . That is, $\mathcal{L}_{ij}^t(\mathcal{H}_i) = \frac{Pr^t(\mathcal{H}_i|i)}{Pr^t(\mathcal{H}_i|j)}$, where $Pr^t(\mathcal{H}_i|j)$ is agent t 's probability assessment of observing \mathcal{H}_i conditional on security j paying out in the second market. According to Bayes' rule, the ratio of agent t 's subjective probabilities conditional on observing \mathcal{H}_i satisfies

$$\frac{\pi'_{it}}{\pi'_{jt}} = \mathcal{L}_{ij}^t(\mathcal{H}_i) \frac{\pi_{it}}{\pi_{jt}}, \quad i, j = 1, \dots, n, \quad (17)$$

where π and π' represent beliefs before and after observing \mathcal{H}_i , respectively. Our next result shows that, whenever agents agree on the 'direction' of the updating, the release of public information about past outcomes decreases the underpricing of favorites and the average overpricing of longshots, that is, the FLB is mitigated. First, we provide the formal notions of agreement and unbiased interpretation of information we use in the result.

Definition 1. Agents *ordinally agree* on the interpretation of \mathcal{H}_i if $\mathcal{L}_{ij}^t(\mathcal{H}_i) < (>) 1$ for some t implies $\mathcal{L}_{ij}^k(\mathcal{H}_i) < (>) 1$ for all k and all i, j . Agents *ordinally agree* if they agree on the interpretation of \mathcal{H}_i for all $i = 1, \dots, n$.

Notice that ordinal agreement is weaker than requiring beliefs to be concordant (Milgrom and Stokey, 1982; Ottaviani and Sorensen, 2012), which would imply ‘cardinal’ agreement, i.e. $\mathcal{L}_{ij}^t = \mathcal{L}_{ij}^k$ for all i, j, t and k .

Definition 2. Agent t is *unbiased* if $\mathcal{L}_{ij}^t(\mathcal{H}_i) > 1$ for all $j \neq i$ and all $i = 1, \dots, n$.

The notion of unbiased beliefs implies that after observing i pay out a trader revises upwards her beliefs about i paying out in the second market relative to all other securities. This leads to both less underpricing of heavy favorites and less overpricing on average of longshots. That is, public information mitigates the FLB, as it is the case in *Figure 5*.

Proposition 1. *If agents are unbiased and ordinally agree then there exists \bar{p} such that for all $p_i > \bar{p}$ the expected price of security i in the second market is higher than in the first market, and the prices of securities $j \neq i$ are lower on average.*

This result implies that when *Theorem 1* holds, new information mitigates the FLB: heavy favorites are less underpriced and longshots exhibit less overpricing on average.

Proof. Fix the state of the world $\mathbf{p} = (p_1, p_2, \dots, p_n)$. For simplicity, we assume that, when indifferent between investing or not in the market, all agents decide to invest –the proof logic would be the same as long as the arrival of public information does not alter agents’ decision to participate when indifferent. Given (17), market shares (i.e. prices) in the second market after observing \mathcal{H}_i are given by

$$\begin{aligned} s_i &= \frac{1}{\int_T w_t dt} \int_T w_t \mathbf{1} \left[\left(\frac{\pi'_{it}}{\pi'_{1t}}, \dots, \frac{\pi'_{it}}{\pi'_{nt}} \right) < \left(\frac{s_i}{s_1}, \dots, \frac{s_i}{s_n} \right) \right] dt \\ &= \frac{1}{\int_T w_t dt} \int_T w_t \mathbf{1} \left[\left(\frac{\pi_{it}}{\pi_{1t}}, \dots, \frac{\pi_{it}}{\pi_{nt}} \right) < \left(\mathcal{L}_{1i}^t(\mathcal{H}_i) \frac{s_i}{s_1}, \dots, \mathcal{L}_{ni}^t(\mathcal{H}_i) \frac{s_i}{s_n} \right) \right] dt. \end{aligned}$$

It is straightforward to check that, when the distribution of prior beliefs is continuous and has full support, the release of public information leads to a higher s_i when $\mathcal{L}_{ij}^t(\mathcal{H}_i) > 1$ for all j and all t : at any given price, the mass of agents that would consider security i the optimal investment has gone up after observing the information.

Next, notice that we must have $Pr(\mathcal{H}_i) \rightarrow 1$ as $p_i \rightarrow 1$. Thus, if agents ordinally agree and are unbiased then $Pr(\mathcal{L}_{ij}^t(\mathcal{H}_i) > 1 \forall j \neq i) \rightarrow 1$ as $p_i \rightarrow 1$, implying that the probability that $\rho_i (= s_i)$ is higher in the second market is close to one for p_i close to one. That is,

for p_i sufficiently high the expected price of security i is higher in the second market than in the first market. This, in turn, implies that $\sum_{j \neq i} \rho_j$ goes down in expectation. \square

C General Approach to Estimating Heterogeneity

We describe here how to conduct step 1 in the estimation of a heterogeneous population model laid out in Section 5, which involves solving the system of equations (7):

$$s_i^k = \frac{1}{\int_T w_t dt} \int_T w_t \mathbf{1} \left[U_t(p_i^k, R_i^k) > U_t(p_j^k, R_j^k) \forall j \neq i \right] dt, \quad i = 1, \dots, n^k.$$

Accordingly, the goal is to characterize U_t so that we can estimate the heterogeneous population model by inverting the system of equations (7) to recover \mathbf{p}^k . To do so, we assume agents have expected utility preferences and exhibit heterogeneity in beliefs and in risk attitudes. Accordingly, agent t 's payoff from investing in security i is given by

$$U_t(p_i, R_i) = \pi_{it} u(w_t R_i, \gamma_t) + (1 - \pi_{it}) u(-w_t, \gamma_t), \quad (18)$$

where π_{it} is agent t 's belief about security i and u is utility over wealth with risk attitudes governed by the one-dimensional parameter γ_t –e.g., the coefficient of absolute risk aversion in CARA utility. The cardinality of expected utility allows to normalize the utility from losing w_t to be zero, i.e., $u(-w_t, \gamma_t) = 0$. In addition, since agent preferences over securities in a given market are invariant to a monotonic transformation of utility, we can take the log of the RHS of (18) and subtract $\log p_1$ to write U_t as

$$U_t(p_i, R_i) = \log u(w_t R_i, \gamma_t) + \xi_i + \nu_{it},$$

where $\xi_i = \log p_i - \log p_1$ and $\nu_{it} = -\log r_{it}$. If we assume that risk parameters and beliefs are independently distributed and that agents have equal endowments (normalized to 1),²¹ we can write (7) as

$$s_i^k = \int_{\gamma_t \in \mathbb{R}} \int_{\nu_t \in \mathbb{R}^n} \mathbf{1} \left[\log u(R_i^k, \gamma_t) + \xi_i + \nu_{it} > \log u(R_j^k, \gamma_t) + \xi_j + \nu_{jt} \forall j \neq i \right] dP(\nu_t; \theta) dH(\gamma_t),$$

for all $i = 1, \dots, n^k$, where $P(\cdot; \theta)$ is the (continuous) distribution of $\nu_t = (\nu_{1t}, \dots, \nu_{n^k t})$ and H represents the distribution of risk parameter γ_t . Thus, as long as this system of market shares has a unique solution in terms of the vector $(\xi_1, \dots, \xi_{n^k})$, we can recover

²¹Alternatively, we could assume $w_t = a > 0$ for all $t \in T$. Generally, without information on endowments, the risk parameter is not separately identified from a .

the underlying probabilities \mathbf{p}^k from observed odds \mathbf{R}^k .

One way to ensure the existence of a unique solution is to assume, as we do in our estimation, that ν_t is a variance mixture of independent logit errors with mixing distribution $G(\sigma)$. Accordingly, market shares are given by

$$s_i^k = \int_{\gamma_t \in \mathbb{R}} \int_{\sigma \in \mathbb{R}_+} \frac{\exp\left(\frac{1}{\sigma} \log u(R_i^k, \gamma_t) + \frac{1}{\sigma} \xi_i\right)}{\sum_{j=1}^{n^k} \exp\left(\frac{1}{\sigma} \log u(R_j^k, \gamma_t) + \frac{1}{\sigma} \xi_j\right)} dG(\sigma) dH(\gamma), \quad i = 1, \dots, n^k. \quad (19)$$

This system has a solution, given that it satisfies the sufficient conditions in Berry (1994), and it is unique as shown by Berry et al. (2012).

D Non-parametric Identification of Belief Heterogeneity

In this section we formally prove identification of the distribution of beliefs, which is the key primitive we estimate in our model. The key source for identification stems from variation in the fundamentals $\mathbf{p} = (p_1, \dots, p_n) \in \text{int}\Delta^{n-1}$ across different markets. In order to focus our attention on the logic of identification, we consider here a simplified setting with two horses and track take $\tau = 0$. Our discussion however easily generalizes to the n -horse context. Indeed, the additional variation made possible by n -horses rather than two horses only aids identification rather than complicating it. Observe that an n -horse can always replicate a two horse race by letting $n - 1$ horses have arbitrarily similar state probabilities but not vice versa, and thus the two horse problem is the essential setting to study.

In each race (i.e. market) we have two horses labeled $i = 1, 2$ and can observe the market share s of horse 1 (the share of horse 2 being simply $s_2 = 1 - s$) and expected returns (ER_1, ER_2) as a function of the market share. The belief heterogeneity model relates these market observables via the random utility

$$u_i = \log ER_1 - \log ER_2 - \epsilon_i$$

where ϵ_i corresponds to the belief disturbance associated with agent i . Agent i in the market bets on horse 1 if $u_i > 0$. Assuming a continuum of agents in the market and ϵ_i distributed independently of (ER_1, ER_2) with distribution function H , we thus have that

$$s = H\left(\log \frac{p}{s} - \log \frac{1-p}{1-s}\right), \quad (20)$$

since $ER_i = \frac{p_i}{s_i}$. We assume H satisfies the usual regularity conditions for a CDF, namely, it is continuous and strictly monotone over the support. Note that, for any $s \in (0, 1)$ there

exists a unique p that solves the above equation. In particular letting $p \rightarrow 1$ we have that $\log \frac{p}{s} \rightarrow 0$ and $\log \frac{1-p}{1-s} \rightarrow -\infty$, and thus the RHS of (20) approaches 1. Likewise as $p \rightarrow 0$ we have that RHS of (20) approaches 0. Because the RHS of (20) is continuous in p , then by the intermediate value theorem for any $s^c \in (0, 1)$ there exists a $p^c \in (0, 1)$ that solves (20) and hence sustains s^c in equilibrium. Thus the support of s under the model is the full unit interval $(0, 1)$ and identification of H becomes straightforward: if we invert both sides of (20) we have

$$H^{-1}(s) = \log \frac{p}{s} - \log \frac{1-p}{1-s}. \quad (21)$$

Because all the terms on the RHS are derived from the data, we thus identify the inverse of H for all $s \in (0, 1)$ and hence H itself.

In the model we estimate in the paper, we restrict H further to be a variance mixture of logistic distributions. That is, for any $t \in \mathbb{R}$ we model

$$H(t) = \int_{\sigma>0} F(t | \sigma) dG(\sigma).$$

where $F(\cdot | \sigma)$ is a standard logistic distribution with scale parameter σ . The object we estimate is G , i.e., the distribution of “types” in the population where each type corresponds to a different variance in beliefs, and G is assumed to have finite support. It is important to emphasize that the support points themselves are not known ex-ante. Hence if we let \mathbb{G} denote the set of all distribution functions with finite support, this is an infinite dimensional space of distributions. Using standard results from the identifiability of logistic mixture models (see Theorem 1 in Shi et al. (2012)), we can recover G from H .

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