On the Possibility of Trade with Pure Common Values under Risk Neutrality

Ricardo Serrano-Padial*
University of Wisconsin-Madison
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Abstract

This paper investigates the existence of bargaining mechanisms that induce trade with positive probability when agents are risk neutral, which constitutes a polar case not covered by existing no trade results. It is shown that a *quasi* no-trade theorem holds in the bilateral case: if the distributions of traders' private signals are continuous, no equilibrium with positive probability of trade exists in any trade environment with pure common values. With discrete distributions trade only occurs when the seller and the buyer receive their lowest and highest signals, respectively. A counterexample in which trade happens with probability one is provided to show that the result fails to hold when there are more than two traders. A property of multilateral mechanisms eliciting trade is that buyers' payments cannot equal expected conditional values almost everywhere. This implies that trade is incompatible with information aggregation in common value environments.

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1 Introduction

Existing no-trade theorems assert that, under strict risk aversion, trade cannot happen with positive probability in pure common value environments when there is uncertainty about the value of the goods traded (Tirole (1982), Milgrom and Stokey

*Email: rserrano@ssc.wisc.edu; Web: www.ssc.wisc.edu/∼rserrano; Address: Department of Economics, Social Science Building, 1180 Observatory Drive, Madison, WI 53706-1393.

(1982), Morris (1994)). However, it is an open question whether risk neutral agents may agree to trade. Under risk neutrality, these no-trade theorems only state that, if the initial allocation is Pareto optimal (i.e. any allocation in a common value setting) and agents receive private information about the value of the goods traded, there is no trade that strictly improves such initial allocation. Hence, they do not rule out the existence of weakly individually rational (IR) and incentive compatible (IC) bargaining mechanisms leading to trade with positive probability. In any such mechanism, agents would be indifferent between the initial and the final allocation, but they could nonetheless decide to trade.

I address this often overlooked indeterminacy by investigating whether trade can happen when (i) risk-neutral traders have common priors about the unknown value of the object to be traded and (ii) some of them receive private signals about such value.[1] I show that, if there is one buyer and one seller, there is no bargaining mechanism in which trade can happen with positive probability when the signal distributions are continuous. When the signal space is finite, trade can only occur when the seller receives her lowest possible signal and the buyer gets her highest signal.

This result does not extend to the multilateral case with more than two traders, as long as at least two of them receive private information. I provide a counterexample with two buyers and one seller in which trade happens with probability one. The key feature of multilateral mechanisms eliciting trade that is absent in any bilateral environment is the possibility to condition the transfer between a buyer and a seller on the signals received by other agents. That is, even in the extreme case in which there are no gains from trade, adverse selection can be mitigated by using private information of agents not directly involved in a given transaction.[2]

An important restriction on multilateral mechanisms eliciting trade is that buyers' payments and expected values (conditional on the vector of traders' signals) cannot be equal almost everywhere.[3] This means that no multilateral bargaining mechanism is *ex post* incentive compatible, i.e. in some transactions one party will regret trading.[4] In addition, it implies that, when we do observe trade, there is a strong violation of the information aggregation properties of prices. Consequently, the *efficient markets hypothesis* (Fama (1970)) for risk neutral traders, which states that prices equal expected asset values conditional on all the available information,

[1] No trade results also apply to the case of heterogenous priors as long as players have concordant beliefs in the sense of Milgrom and Stokey (1982) and the initial allocation is Pareto optimal. By restricting attention to the common prior case, I look at a class of trade environments in which all the initial allocations are optimal.

[2] I thank Joel Sobel for pointing out this fact by suggesting the use of a third party's signals to induce trade between a buyer and a seller.

[3] An exception to this rule exists when trade only takes place among lowest signal sellers and highest signal buyers.

[4] For instance, in the example provided, when all traders receive the highest possible signal, buyers pay the seller more than the conditional expected value of the object.

either violates incentive compatibility or requires zero trade volume.

2 A Pure Common Value Environment with Private Information

There are n risk neutral sellers, each of them owning one unit of an indivisible object, and m risk neutral buyers, who can buy at most one unit. The unknown value of the object is given by V, with probability distribution G. The support of G is denoted by $\mathcal{V}$. Each individual i receives a private signal stochastically related to V, $S_i \sim F_i(.|v)$.[5] Let $\mathcal{S}_i$ be the support of $F_i(.|v)$.

I make the following assumptions.

Assumption 1 *$\mathcal{V} \subset \mathbb{R}_+$ is compact and has at least two elements. $\mathcal{S}_i$ are compact for $i = 1, ..., n+m$, and there exists at least an agent j for whom $\mathcal{S}_j$ has more than one element.*

Compactness is not essential, while the minimum number of elements in the signal supports is necessary for the existence of at least two agents with distinct posterior probabilities about V. Notice that I allow for asymmetries in the quality of information by letting the signal distributions to differ across agents.

Assumption 2 *G and F_i are common knowledge, $i = 1, ..., n+m$.*

Assumption 3 *G has full support. In addition, for any $v \in \mathcal{V}$, the conditional distribution of $S = (S_1, S_2, ..., S_{n+m})$, denoted $F(.|v)$, has full support.*

This implies that the conditional distribution of S_i with respect to any vector of the other traders' signals $s_{\text{-}i} \in \mathcal{S}_{\text{-}i}$ also has full support. The next assumption establishes the stochastic relationship between values and signals. It roughly states that higher signals are more likely when the value is high and viceversa.

Assumption 4 (MLRP) *For all i, $F_i(.|v)$ satisfies the strict monotone likelihood ratio property: $\frac{f_i(s_i|v)}{f_i(s_i'|v)} > \frac{f_i(s_i|v')}{f_i(s_i'|v')}$ for all $s_i, s_i' \in \mathcal{S}_i$ such that $s_i > s_i'$ and all $v, v' \in \mathcal{V}$ such that $v > v'$.*

Overall, the above assumptions lead to the strict monotonicity of the expected value of the object conditional on agents' signals:

$$\mathbb{E}(v|s) > \mathbb{E}(v|s'), \tag{1}$$

for all $s, s' \in \mathcal{S} = \prod_i \mathcal{S}_i$ such that $s > s'$.

Assumption 5 (Common values) *Signals are payoff irrelevant, i.e. agents' utility is only a function of V.*

[5] In what follows, I use uppercase letters to denote random variables (V, S_i) or cumulative distribution functions(G, F) and lowercase to denote realizations of random variables (v, s_i) or probabilities and densities (g, f).

3 Trade Mechanisms

Direct bargaining mechanisms specify, for every signal profile $s \in \mathcal{S}$ reported (truthfully) by the agents, both a payment vector and a vector of probabilities of trading the object.[6] The sum of payments is zero (balanced budget) and the sum of sellers' probabilities is equal to the sum of buyers' probabilities. I denote the payment function $x\colon \mathcal{S} \to \mathbb{R}^{n+m}$, with $\sum_i x_i(s) = 0$ for all $s \in \mathcal{S}$, and the vector of trade probabilities $q\colon \mathcal{S} \to [0,1]^{n+m}$, which satisfies $\sum_{i=1}^{n} q_i(s) = \sum_{i=n+1}^{n+m} q_i(s) \leq n$ for all $s \in \mathcal{S}$. Abusing notation, I use $v(s)$ to refer to $\mathbb{E}(V|s)$. Given the above assumptions, the expected (interim) payoffs in mechanism (q,x) for sellers and buyers are, respectively,[7]

$$\begin{aligned}\pi_i(s_i) :=\ & \mathbb{E}_{s_{\text{-}i}}(-x_i(s_i, S_{\text{-}i}) - q_i(s_i, S_{\text{-}i})v(s_i, S_{\text{-}i})|s_i) \\ =\ & \int_{s_{\text{-}i}\in\mathcal{S}_{\text{-}i}} \big\{-x_i(s_i, s_{\text{-}i}) - q_i(s_i, s_{\text{-}i})v(s_i, s_{\text{-}i})\big\} dF_{\text{-}i}(s_{\text{-}i}|s_i) \end{aligned} \tag{2}$$

and

$$\begin{aligned}\pi_j(s_j) :=\ & \mathbb{E}_{s_{\text{-}j}}(q_j(s_j, S_{\text{-}j})v(s_j, S_{\text{-}j}) - x_j(s_j, S_{\text{-}j})|s_j) \\ =\ & \int_{s_{\text{-}j}\in\mathcal{S}_{\text{-}j}} \big\{q_j(s_j, s_{\text{-}j})v(s_j, s_{\text{-}j}) - x_j(s_j, s_{\text{-}j})\big\} dF_{\text{-}j}(s_{\text{-}j}|s_j). \end{aligned} \tag{3}$$

The mechanism (q,x) is *individually rational* (IR) if $\pi_i^s(s_i) \geq 0$ for all $s_i \in \mathcal{S}_i$, $i = 1, ..., n$ and $\pi_j^b(s_j) \geq 0$ for all $s_j \in \mathcal{S}_j$, $j = n+1, ..., n+m$, i.e.

$$-\mathbb{E}_{s_{\text{-}i}}(x_i(s_i, S_{\text{-}i})|s_i) \geq \mathbb{E}_{s_{\text{-}i}}(q_i(s_i, S_{\text{-}i})v(s_i, S_{\text{-}i})|s_i), \tag{4}$$

and

$$\mathbb{E}_{s_{\text{-}j}}(x_j(s_j, S_{\text{-}j})|s_j) \leq \mathbb{E}_{s_{\text{-}j}}(q_j(s_j, S_{\text{-}j})v(s_j, S_{\text{-}j})|s_j). \tag{5}$$

In addition, (q,x) is *incentive compatible* (IC) if telling the truth is a Bayesian Nash equilibrium:

$$\begin{aligned}&\mathbb{E}_{s_{\text{-}i}}(x_i(s_i', S_{\text{-}i}) + q_i(s_i', S_{\text{-}i})v(s_i', S_{\text{-}i})|s_i') \leq \\ &\qquad \mathbb{E}_{s_{\text{-}i}}(x_i(s_i, S_{\text{-}i}) + q_i(s_i, S_{\text{-}i})v(s_i', S_{\text{-}i})|s_i')\end{aligned} \tag{6}$$

for all $s_i, s_i' \in \mathcal{S}_i$, $i = 1, ..., n$ and

$$\begin{aligned}&\mathbb{E}_{s_{\text{-}j}}(x_j(s_j', S_{\text{-}j}) - q_j(s_j', S_{\text{-}j})v(s_j', S_{\text{-}j})|s_j') \leq \\ &\qquad \mathbb{E}_{s_{\text{-}j}}(x_j(s_j, S_{\text{-}j}) - q_j(s_j, S_{\text{-}j})v(s_j', S_{\text{-}j})|s_j')\end{aligned} \tag{7}$$

for all $s_j, s_j' \in \mathcal{S}_j$, $j = n+1, ..., n+m$.

[6] Myerson and Satterthwaite (1983) analyze bilateral bargaining mechanisms in the context of trade with pure private values.

[7] I abuse notation again by denoting $F_{\text{-}i}(.|s_i)$ the distribution of all agents' signals except agent i's ($S_{\text{-}i}$) conditional on agent i's signal, $i = 1, ..., n+m$.

4 Bilateral Case: A *No-Trade* Theorem

I analyze the possibility of trade with one buyer and one seller under two different scenarios: the continuous case ($F_{\text{-}i}(.|s_i)$ is absolutely continuous, $i = 1, 2$) and the finite case. I show that in the former there is no mechanism (q, x) involving positive probability of trade that is individually rational and incentive compatible. If there is no such mechanism, using the *revelation principle* we can assert that there is no equilibrium with positive probability in any trade environment (with voluntary participation) satisfying *Assumptions* 1-5 and the absolute continuity condition.

However, when the signal space is finite, equilibria with trade exist, although trade is restricted to take place only when both the seller receives her lowest possible signal ($\underline{s}_1$) and the buyer gets her highest signal ($\overline{s}_2$).

I simplify notation by denoting $x(s_1, s_2)$ the buyer's payment to the seller, $x(s_1, s_2) := -x_1(s_1, s_2) = x_2(s_1, s_2)$, and $q(s_1, s_2)$ the buyer's probability of getting the object, $q(s_1, s_2) := q_1(s_1, s_2) = q_2(s_1, s_2)$. The proof is in the *Appendix.*

Theorem 1 *(a) (**Continuous Case**) If $F_1(.|s_2)$ and $F_2(.|s_1)$ are absolutely continuous with densities $f_{\text{-}i}(s_{\text{-}i}|s_i) > \eta > 0$ ($i = 1, 2$) for all $s \in \mathcal{S}$, there is no IR and IC bargaining mechanism involving positive probability of trade in any bilateral trade environment satisfying Assumptions 1-5.*

*(b) (**Finite Case**) Assume $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is finite. The only IR and IC bargaining mechanisms satisfying Assumptions 1-5 that exist involve no trade for all $s \in \mathcal{S} \setminus \{(\underline{s}_1, \overline{s}_2)\}$.*

The proof of *Theorem 1* consists of two parts. First, I show that the zero sum game nature of the trading environment implies that IR constraints hold with equality almost surely. Given this, if there is trade with positive probability for some seller's signal $s_1 > \underline{s}_1$ and for some buyer's signal $s_2 < \overline{s}_2$, in order to satisfy seller's IC constraints we need to violate buyer's IC constraints. This result is driven by the strict monotonicity of $v(s_1, s_2)$. To see how, assume that the seller gets $\underline{s}_1$ and that IR holds with equality for $\underline{s}_1$. For the seller's IC constraint under $\underline{s}_1$ to be satisfied, the expected payment under $s_1 > \underline{s}_1$ has to be strictly lower than the expected net value of the object ($q(s_1, s_2)v(s_1, s_2)$), when expectations (taken over buyer's signals) are conditional on $\underline{s}_1$. Otherwise, by strict monotonicity, it will be profitable for the seller to lie and report $s_1 > \underline{s}_1$.[8] This implies that q and x, on average, favor the buyer when low signals are more likely. But given this, a buyer with a very high signal has an incentive to report a low signal, provided that, by strict monotonicity, the object is more valuable to him than to a buyer receiving the

[8] Strict monotonicity implies that $\mathbb{E}_{s_2}(q(s_1, s_2)v(\underline{s}_1, s_2)|\underline{s}_1) < \mathbb{E}_{s_2}(q(s_1, s_2)v(s_1, s_2)|\underline{s}_1)$. Hence, if the last term is not strictly bigger than $\mathbb{E}_{s_2}(x(s_1, s_2)|\underline{s}_1)$, a seller receiving $\underline{s}_1$ will get a strictly positive payoff from reporting s_1.

lower (reported) signal. Hence, the scenario in which positive probability of trading does not violate IR and IC constraints involves $q(s) > 0$ only for $s = (\underline{s}_1, \overline{s}_2)$, which is a non-null event in the finite case by the full support assumption.

In sum, the strict monotonicity of $v(s_1, s_2)$, coupled with IR constraints holding with equality, forces net values to be bigger than payments for low signal profiles in order to satisfy low-signal seller's IC constraints, but this provides incentives for high-signal buyers to lie. Thus, the only way all constraints are satisfied is when net values and payments are zero (except maybe for $(\underline{s}_1, \overline{s}_2)$).

5 The Multilateral Case

A no-trade result like *Theorem 1* does not exist for the case with three or more traders, except when only one trader receives more than one signal.[9] In fact, there are mechanisms (q, x) leading to trade with probability one. However, except in boundary cases, to elicit trade the probability that payments equal values needs to be strictly less than one (*Theorem 2*). As an example, consider the following trade environment.

Example 1 *There are three traders, one seller ($n = 1$) and two buyers ($m = 2$). Each of the traders receives a private signal $s_i \in \{0, 1\}$, with $\mathbb{P}(s_i = 0) = \mathbb{P}(s_i = 1) = \frac{1}{2}$. The conditional distribution of s_{-i} given s_i is given by*

$$\mathbb{P}(s_{-i}|s_i) = \begin{cases} \frac{1}{2} & \text{if } s_{-i} = (0,0),\ s_i = 0 \\ \frac{1}{8} & \text{if } s_{-i} = (0,0),\ s_i = 1 \\ \frac{3}{16} & \text{if } s_{-i} \in \{(0,1),(1,0)\} \\ \frac{1}{8} & \text{if } s_{-i} = (1,1),\ s_i = 0 \\ \frac{1}{2} & \text{if } s_{-i} = (1,1),\ s_i = 1, \end{cases} \tag{8}$$

and the expected value of the asset conditional on the vector of signals is

$$v(s) = \begin{cases} 0 & \text{if } s = (0,0,0) \\ \frac{1}{4} & \text{if } \sum s_i = 1 \\ \frac{3}{4} & \text{if } \sum s_i = 2 \\ 1 & \text{if } s = (1,1,1) \end{cases} \tag{9}$$

In this setting, it is possible to find a direct bargaining mechanism (x, q) such that the object is traded for all s. In particular, we can find a payment function

[9] In such a case, it is easy to show that in the finite case trade can only happen when the trader with a signal space that is not a singleton either receives the lowest possible signal (seller) or the highest possible one (buyer).

$x(s)$ such that the probability of trade for buyer i is

$$q_i(s) = \begin{cases} 0 & \text{if } s_i < s_j \\ \frac{1}{2} & \text{if } s_i = s_j \\ 1 & \text{if } s_i > s_j, \end{cases} \tag{10}$$

where j denotes the other buyer. The above probabilities mean that the buyer with the highest signal receives the object for sure except when both buyers have the same signal, in which case each buyer receives the object with probability one half.[10]

The IR constraints (4)-(5) in this zero-sum environment hold with equality.[11] Thus, the IR constraints for buyer $i \in \{2, 3\}$ given (8)-(10) are, respectively,[12]

$$\frac{1}{2}x_i(0,(0,0)) + \frac{3}{16}[x_i(0,(1,0)) + x_i(0,(0,1))] + \frac{1}{8}x_i(0,(1,1)) = \frac{3}{128}$$

and

$$\frac{1}{8}x_i(1,(0,0)) + \frac{3}{16}[x_i(1,(1,0)) + x_i(1,(0,1))] + \frac{1}{2}x_i(1,(1,1)) = \frac{63}{128}.$$

Let $\Sigma x_i(s_i, s_{-i}) = x_2(s_2,(s_1,s_3)) + x_3(s_3,(s_1,s_2))$. The IR constraints for the seller are given by

$$\frac{1}{2}\Sigma x_i(0,(0,0)) + \frac{3}{16}[\Sigma x_i(1,(0,0)) + \Sigma x_i(0,(0,1))] + \frac{1}{8}\Sigma x_i(1,(0,1)) = \frac{3}{16}$$

and

$$\frac{1}{8}\Sigma x_i(0,(1,0)) + \frac{3}{16}[\Sigma x_i(1,(1,0)) + \Sigma x_i(0,(1,1))] + \frac{1}{2}\Sigma x_i(1,(1,1)) = \frac{13}{16}.$$

Given that the IR constraints hold with equality, the IC constraints (6) for buyer i reduce to the following inequalities:

$$\frac{1}{2}x_i(1,(0,0)) + \frac{3}{16}[x_i(1,(1,0)) + x_i(1,(0,1))] + \frac{1}{8}x_i(1,(1,1)) \geq \frac{15}{128}$$

and

$$\frac{1}{8}x_i(0,(0,0)) + \frac{3}{16}[x_i(0,(1,0)) + x_i(0,(0,1))] + \frac{1}{2}x_i(0,(1,1)) \geq \frac{11}{128}.$$

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[10] These trade probabilities are similar to what happens in symmetric equilibria of common value auctions.

[11] This is shown in the proof of *Theorem 2*.

[12] Recall that the notation followed is $x_i(s_i, s_{-i})$. For instance, $x_i(0,(1,0))$ denotes the payment from buyer i to the seller when the signals received are zero for both buyers one for the seller.

Similarly, the IC constraints for the seller are

$$\frac{1}{2}\Sigma x_i(0,(1,0)) + \frac{3}{16}[\Sigma x_i(1,(1,0)) + \Sigma x_i(0,(1,1))] + \frac{1}{8}\Sigma x_i(1,(1,1)) \leq \frac{3}{16}$$

and

$$\frac{1}{8}\Sigma x_i(0,(0,0)) + \frac{3}{16}[\Sigma x_i(1,(0,0)) + \Sigma x_i(0,(0,1))] + \frac{1}{2}\Sigma x_i(1,(0,1)) \leq \frac{13}{16}.$$

If we further require that a buyer does not pay when he does not receive the object with positive probability ($x_i(0,(.,1)) = 0$ for $i = 2,3$),[13] then

(i) the payments when the object is (in expectation) *least* valuable are strictly less than the value: $x_i(0,(0,0)) < v(0,0,0) = 0$ for $i = 2,3$,[14]

(ii) for some buyer i, the payment when the object is *most* valuable is larger than the value: $x_i(1,(1,1)) > v(1,1,1) = 1$,[15] and

(iii) for some buyer i, $x_i(1,(1,0)) < v(0,0,0)$.

An example of such a mechanism (x,q) is given by (10) and the following symmetric payment function:

$$x_i(s_i, s_{-i}) = \begin{cases} -\frac{3}{16} & \text{if } s_i = 0,\ s_{-i} = (0,0) \\ \frac{5}{8} & \text{if } s_i = 0,\ s_{-i} = (1,0) \\ 0 & \text{if } s_i = 0,\ s_{-i} = (s_1,1) \\ \frac{3}{4} & \text{if } s_i = 1,\ s_{-i} = (0,0) \\ -\frac{5}{2} & \text{if } s_i = 1,\ s_{-i} = (1,0) \\ \frac{3}{8} & \text{if } s_i = 1,\ s_{-i} = (0,1) \\ \frac{51}{32} & \text{if } s_i = 1,\ s_{-i} = (1,1). \end{cases} \tag{11}$$

As illustrated by this example, incentive compatibility prevents payments to be equal to values almost everywhere, except if buyers (sellers) only trade when they receive their highest (lowest) signal. This is formally stated in the following theorem. Let $q_i(s_i) = \int_{s_{\text{-}i} \in \mathcal{S}_{\text{-}i}} q_i(s_i, s_{\text{-}i}) dF_{\text{-}i}(s_{\text{-}i}|s_i)$.

Theorem 2 *If (x,q) satisfies Assumptions 1-5, then $\mathbb{P}(-x_i(s_i,S_{-i}) = v(s_i,S_{-i})|q_i(s_i) > 0, s_i > \underline{s}_i) < 1$ for all $i \in \{1,...,n\}$ and $\mathbb{P}(x_j(s_j,S_{-j}) = v(x_j(s_j,S_{-j})|q_j(s_j) > 0, s_j < \overline{s}_j) < 1$ for all $j \in \{n+1,...,n+m\}$.*

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[13] This is a common feature in many trade environments, such as double auctions.

[14] This is easy to check by subtracting the buyers' first IR constraint from the second IC constraint and setting $x_i(0,(1,1)) = 0$.

[15] This is due to the restrictions that (i) and the seller's second IR constraint and first IC constraint impose on $\Sigma x_i(1,(1,1))$.

Proof. First, it is straightforward to show that IR constraints hold with equality almost surely (henceforth *a.s.*). Notice that pure common values plus the requirement that $\sum_i x_i(s) = 0$ and $\sum_{i=1}^{n} q_i(s) = \sum_{i=n+1}^{n+m} q_i(s)$ for all $s \in \mathcal{S}$ imply that

$$\mathbb{E}\left(-\sum_{i=1}^{n+m} x_i(S) - v(S)\left[\sum_{i=1}^{n} q_i(S) - \sum_{i=n+1}^{n+m} q_i(S)\right]\right) = \sum_{i=1}^{n+m} \mathbb{E}(\pi_i(S)) = 0.$$

By *Assumption 3* (full support) this is only true if $\pi_i(s_i) = 0$ *a.s.*, $i = 1, ..., n+m$, provided that $\pi_i(s_i)$ is nonnegative by IR.

Now assume that $\mathbb{P}(-x_i(s_i, S_{-i}) = v(S)|q_i(s_i) > 0) = 1$ for some seller i and signal $s_i > \underline{s}_i$. By strict monotonicity $v(s_i', s_{-i}) < v(s_i, s_{-i})$ for all $s_i' < s_i$ and all s_{-i}. But then, if $q_i(s_i) > 0$ she can earn a strictly positive payoff by reporting s_i when her true signal is s_i', thus violating incentive compatibility. A symmetric argument holds for buyers. ■

Appendix

Proof of *Theorem 1.* Denote $A_1 \subseteq \mathcal{S}_1$ and $A_2 \subseteq \mathcal{S}_2$ the sets for which (4) and (5) hold with equality, respectively. Accordingly, IC constraints simplify to

$$\mathbb{E}_{s_2}(x(s_1, S_2)|s_1') \leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1', S_2)|s_1') \tag{12}$$

for all $s_1' \in A_1$ and all $s_1 \in \mathcal{S}_1$, and

$$\mathbb{E}_{s_1}(x(S_1, s_2)|s_2') \geq \mathbb{E}_{s_1}(q(S_1, s_2)v(S_1, s_2')|s_2') \tag{13}$$

for all $s_2' \in A_2$ and all $s_2 \in \mathcal{S}_2$.

PART (a): the continuous case

Assume trade occurs with positive probability, i.e. the sets $\mathcal{S}_i^* = \{s_i \in \mathcal{S}_i\colon\ \mathbb{E}_{s_{\text{-}i}}(q(s_i, S_{\text{-}i})|s_i) > 0\}$ are non-null, $i = 1, 2$. By (12) and the strict monotonicity of $v(\,.\,,.)$ we have that, for all $s_1 > s_1'$ with $s_1 \in \mathcal{S}_1^*$ and $s_1' \in A_1$,

$$\begin{aligned} \mathbb{E}_{s_2}(x(s_1, S_2)|s_1') &\leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1', S_2)|s_1') \\ &< \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1, S_2)|s_1'). \end{aligned} \tag{14}$$

These inequalities lead to the following result, whose validity we assume for the moment.

Claim 1 *If trade occurs with positive probability there exists a small enough signal* $s_1' \in A_1$ *such that, for all* $s_2' \in A_2$,

$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(x(S_1,S_2)|s_1')|s_2'] < \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1,S_2)v(S_1,S_2)|s_1')|s_2']. \tag{15}$$

Note that $F_1(.|s_2') \times F_2(.|s_1')$ induces a well defined product measure on $\sigma(\mathcal{S}_1 \times \mathcal{S}_2)$, the σ-field generated by $\mathcal{S}_1 \times \mathcal{S}_2$. In addition, it is easy to check that both $x(S_1,S_2)$ and $q(S_1,S_2)v(S_1,S_2)$ are integrable with respect to this product measure.[16] Therefore, we can apply Fubini's theorem and switch the order of integration on both sides of (15):

$$\mathbb{E}_{s_2}[\mathbb{E}_{s_1}(x(S_1,S_2)|s_2')|s_1'] < \mathbb{E}_{s_2}[\mathbb{E}_{s_1}(q(S_1,S_2)v(S_1,S_2)|s_2')|s_1'] \tag{16}$$

Given that $\{s_2 \in A_2\}$ is a probability one event and that $\mathcal{S}_2^*$ is non-null, this strict inequality implies that there exist $s_2 \in \mathcal{S}_2^*$ and a high enough signal $s_2' \in A_2$ satisfying $s_2 < s_2'$ such that

$$\mathbb{E}_{s_1}(x(S_1,s_2)|s_2') < \mathbb{E}_{s_1}(q(S_1,s_2)v(S_1,s_2)|s_2') \leq \mathbb{E}_{s_1}(q(S_1,s_2)v(S_1,s_2')|s_2')$$

where the last inequality is due to the strict monotonicity of $v(\,.\,,.)$.

But since (13) holds for all $s_2' \in A_2$, the buyer's IC constraint for s_2' is violated. Hence, the only mechanism that satisfies IR and IC constraints involves $q(s_1,s_2)$ equal to zero *a.s.*

PART (b): the finite case

By *Assumption 3* and the finiteness of $\mathcal{S}$, all $s \in \mathcal{S}$ occur with positive probability. Hence, IR constraints hold with equality and (12)-(13) are satisfied for all seller and buyer's signals.

First, I show that $q(s_1,s_2)$ can not be greater than zero for more than one $s \in \mathcal{S}$. Assume that there exist two seller's signals with positive probability of trade. In this case, (15) is satisfied for $s_1' = \underline{s}_1$. Applying Fubini's theorem for $s_2' = \bar{s}_2$ we have that

$$\mathbb{E}_{s_2}[\mathbb{E}_{s_1}(x(S_1,S_2)|\bar{s}_2)|\underline{s}_1] < \mathbb{E}_{s_2}[\mathbb{E}_{s_1}(q(S_1,S_2)v(S_1,S_2)|\bar{s}_2)|\underline{s}_1]$$

Note that for this inequality to hold, there needs to exist a signal $s_2 \in \mathcal{S}_2$ such that

$$\mathbb{E}_{s_1}(x(S_1,s_2)|\bar{s}_2) < \mathbb{E}_{s_1}(q(S_1,s_2)v(S_1,s_2)|\bar{s}_2) \leq \mathbb{E}_{s_1}(q(S_1,s_2)v(S_1,\bar{s}_2)|\bar{s}_2).$$

If this inequality holds for $s_2 \neq \bar{s}_2$ the buyer has an incentive to report untruthfully whenever he receives $\bar{s}_2$, thus violating (13). On the other hand, if the only

[16] The latter is integrable since both $q(.,.)$ and $v(\,.\,,.)$ are bounded. Given this and that $\mathbb{E}_{s_2}(x(s_1,S_2)|\underline{s}_1) \leq \mathbb{E}_{s_2}(q(s_1,S_2)v(s_1,S_2)|\underline{s}_1)$ holds for all $s_1 \in \mathcal{S}_1$ the former is also integrable.

signal for which it holds is $\bar{s}_2$, then it would imply a violation of the IR constraint, since this constraint holds with equality. A similar argument applies when there exist two buyer's signals with positive probability of trade.

Second, I show that when the only $s \in \mathcal{S}$ for which $q(s_1, s_2)$ can be greater than zero is $(\underline{s}_1, \overline{s}_2)$. Assume there exists $s_1 > \underline{s}_1$ with $q(s_1, s_2) > 0$ for some $s_2 \in \mathcal{S}_2$. Then,

$$x(s_1, s_2) = q(s_1, s_2)v(s_1, s_2) > q(s_1, s_2)v(\underline{s}_1, s_2),$$

which violates seller's IC constraint for $\underline{s}_1$. A similar argument applies to any $s_2 < \overline{s}_1$ with $q(s_1, s_2) > 0$ for some $s_1 \in \mathcal{S}_1$.

Finally, it is straightforward to see that any mechanism such that $q(s) = a1_{\{s=(\underline{s}_1,\overline{s}_2)\}}$ and $x(s) = q(s)v(s)$, with $a \in (0, 1]$ satisfies IR constraints and (12)-(13).

Proof of *Claim 1.* First, note that $q(s_1, s_2) = 0$ *a.s.* for any $s_1 \in \mathcal{S}_1 \backslash \mathcal{S}_1^*$, provided $\mathbb{E}_{s_2}(q(s_1, S_2)|s_1) = 0$ and $q(s_1, s_2) \geq 0$. This, in conjunction with (12) and the absolute continuity of $F_2(.|s_1')$, implies that

$$\mathbb{E}_{s_2}(x(s_1, S_2)|s_1') \leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1', S_2)|s_1') = 0$$

a.s. for all $s_1 \in \mathcal{S}_1 \backslash \mathcal{S}_1^*$. Hence, we have that the left hand side of (15) satisfies

$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(x(S_1, S_2)|s_1')|s_2'] \leq \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(x(S_1, S_2)|s_1')[1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}} + 1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}]|s_2']. \quad (17)$$

Likewise,

$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)v(S_1, S_2)|s_1')|s_2'] = \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)v(S_1, S_2)|s_1')[1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}} + 1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}]|s_2']. \quad (18)$$

We have to show that there exists a small enough $s_1' \in A_1$ for which the right hand side of (17) is strictly smaller than the right hand side of (18) or, alternatively, that

$$\overbrace{\mathbb{E}_{s_1}[\mathbb{E}_{s_2}([x(S_1, S_2) - q(S_1, S_2)v(S_1, S_2)]|s_1')1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2']}^{\varphi(s_1',s_2')} < \underbrace{\mathbb{E}_{s_1}[\mathbb{E}_{s_2}([q(S_1, S_2)v(S_1, S_2) - x(S_1, S_2)]|s_1')1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2']}_{\psi(s_1',s_2')}. \quad (19)$$

First, notice that by (12),

$$\begin{aligned} \varphi(s_1', s_2') &\leq \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)[v(s_1', S_2) - v(S_1, S_2)]|s_1')1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2'] \\ &\leq (\overline{v} - \underline{v})\mathbb{P}(\{S_1 < s_1'\} \cap \mathcal{S}_1^*|s_2') =: \underline{\varphi}(s_1', s_2'), \end{aligned} \quad (20)$$

where $\overline{v}$ and $\underline{v}$ denote the maximum and minimum values in $\mathcal{V}$, respectively.

By *Assumption 1* $(\overline{v} - \underline{v})$ is bounded. In addition, the absolute continuity of $F_1(.|s_2')$ implies that $\mathbb{P}(\mathcal{S}_1^* \cap (\mathcal{S}_1 \backslash A_1)|s_2') = 0$. Hence, by arguing again the absolute continuity of $F_1(.|s_2')$, given any $\varepsilon > 0$ we can find a small enough $s_1' \in A_1$ such that $\mathbb{P}(\{S_1 < s_1'\} \cap \mathcal{S}_1^*|s_2') < \varepsilon$. On the other hand, we have that

$$\begin{aligned}
\psi(s_1', s_2') &\geq \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)[v(S_1, S_2) - v(s_1', S_2)]|s_1')1_{\{\{S_1 > s_1'\} \cap \mathcal{S}_1^*\}}|s_2'] \\
&= \int\limits_{\substack{s_1 > s_1', \\ s_1 \in \mathcal{S}_1^*}} \Big[\int\limits_{s_2 \in \mathcal{S}_2} q(s_1, s_2)[v(s_1, s_2) - v(s_1', s_2)] f_2(s_2|s_1') ds_2 \Big] f_1(s_1|s_2') ds_1 \\
&\geq \int\limits_{\substack{s_1 > s_1', \\ s_1 \in \mathcal{S}_1^*}} \Big[\int\limits_{s_2 \in \mathcal{S}_2} q(s_1, s_2)[v(s_1, s_2) - v(s_1', s_2)] \eta^2 ds_2 \Big] ds_1 =: \underline{\psi}(s_1', s_2').
\end{aligned}$$

The inner integral is strictly positive for any $s_1 \in \mathcal{S}_1^*$ such that $s_1 > s_1'$ by strict monotonicity of $v(\,.\,,.)$. In addition, since $\mathbb{P}(\mathcal{S}_1^* \cap A_1|s_2') = \mathbb{P}(\mathcal{S}_1^*|s_2') > 0$, by the absolute continuity of $F_1(.|s_2')$ there is a small enough $s_1' \in A_1$ such that the set of s_1 over which we integrate $(\{S_1 > s_1'\} \cap \mathcal{S}_1^*)$ has positive Lebesgue measure, implying that $\underline{\psi}(s_1', s_2') > 0$. Moreover, $\underline{\psi}(s_1', s_2')$ gets larger as s_1' gets smaller given that both the inner integral and $\{S_1 > s_1'\} \cap \mathcal{S}_1^*$ are bigger for smaller s_1'. Therefore, we can find a small enough $s_1' \in A_1$ such that

$$\varphi(s_1', s_2') \leq \underline{\varphi}(s_1', s_2') < \underline{\psi}(s_1', s_2') \leq \psi(s_1', s_2'). \ \blacksquare$$

This completes the proof of *Theorem 1*. ■

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