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Journal of Economic Theory 147 (2012) 1882–1912

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

# Naive traders and mispricing in prediction markets ☆

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Received 12 May 2010; final version received 31 January 2012; accepted 31 January 2012

Available online 22 May 2012

**Abstract**

This paper studies pricing patterns in a speculative market with asymmetric information populated by both sophisticated and naive traders. Three pricing regimes arise in equilibrium: perfect pricing, with prices equalling asset values, partial mispricing and complete mispricing. Perfect pricing obtains when the presence of naive traders is small although not necessarily zero. When the fraction of naive traders is moderate prices are correct for some values but not for others. Finally, complete mispricing typically arises when the presence of naive traders is sufficiently high. Mispricing exhibits a systematic pattern of overpricing low values and underpricing high values.


*JEL classification:* C72; D44; D82

*Keywords:* Naive traders; Double auction; Common values; Private information; Prediction markets; Favorite-longshot bias

☆ Earlier versions were circulated under various titles. I am indebted to Vince Crawford and David Miller for their generous support and advice, and, especially, to Joel Sobel for his guidance throughout the life of this project. I also thank Tolga Cenesizoglu, Steven Durlauf, Nagore Iriberri, Navin Kartik, Stephen Morris, Jason Murray, Dan Quint, Bill Sandholm, Adam Sanjurjo, John Smith, Xavier Vives and seminar participants at various universities for their helpful suggestions. Support from a IHS Fellowship is gratefully acknowledged. I am grateful to Christian Hellwig (the editor) and three anonymous referees for comments that greatly improved the paper.

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0022-0531/$ – see front matter 
http://dx.doi.org/10.1016/j.jet.2012.05.020

## 1. Introduction

It is hard to overstate the informational role that market prices play in modern economies. Beyond the information conveyed to market participants, it is becoming increasingly common for policy makers, news organizations and the public at large to resort to prices for information about future events. For instance, prices of sovereign bonds are closely tracked to gauge the probability that a country will default on its debt. When discussing the rationale for alternative energy policies oil prices take center stage since they may convey information about oil's future availability.

Prediction Markets epitomize this idea. These are asset markets designed with the sole purpose of forecasting future events, without other considerations in mind such as risk sharing (bonds) or resource allocation (oil futures).[1] The theoretical underpinning behind them is the efficient market hypothesis (Hayek [6], Fama [3]): if traders are perfectly rational, prices in competitive markets convey all the relevant information about the object of trade. Under a strong version of this hypothesis, namely that prices equal expected monetary returns, securities can be designed in a way that their price can be interpreted as a forecast. For instance, the price of an Arrow–Debreu security paying one dollar if some event happens (and zero otherwise) can be seen as an estimate of the event probability, since the latter coincides with the expected monetary return of the security.

At the same time, there is a growing literature documenting departures from perfect rationality and, in the context of these markets, the presence of two types of agents has been empirically observed: sophisticated and naive traders.[2] In light of this evidence, it is important to study the relationship between prices and security returns when we relax the assumption that all market participants are perfectly rational. It is of particular interest to understand whether the presence of naive traders leads to a divergence of prices from returns and, if so, whether the mispricing exhibits a systematic pattern. This analysis may shed light on the conditions needed for prediction markets to produce efficient, unbiased forecasts. In addition, it may help identify ways for outside observers to detect mispricing, for instance, by making public some market information beyond prices.

In this paper, I theoretically address these questions by looking at asset markets in which traders are risk neutral and the ex-post value of the security is the same for all traders. This common values approach implies that the market plays no risk sharing or allocative role, which it is arguably the case in prediction markets. The model has three defining features: (i) the agent population consists of two types, naive and sophisticated, with naive traders following fixed bidding strategies independent of others' equilibrium behavior; (ii) agents hold private information about the value of the security traded in the market; and (iii) the trading mechanism is a two-sided auction in which agents attach to each buy/sell offer a reservation price or *bid*. This mechanism is the most frequently used in existing prediction markets. I focus on large economies and characterize

[1] Examples include the Iowa Electronic Markets (IEM) for presidential elections; the markets for political and economic events in http://www.intrade.com; Google's corporate prediction markets to predict company's performance and future technology trends; and the Hollywood Stock Exchange – a virtual currency market aimed at forecasting movie ticket sales. I refer the reader to Wolfers and Zitzewitz [18] for an overview of prediction markets.

[2] Using data from the IEM 1988 presidential election market, Forsythe et al. [4] find evidence of this typology: some traders exhibited behavioral or ideological biases (i.e. party affiliation influenced their trading behavior) and earned negative average returns while there were also "marginal traders" who did not exhibit such biases and earned a 10% average return.

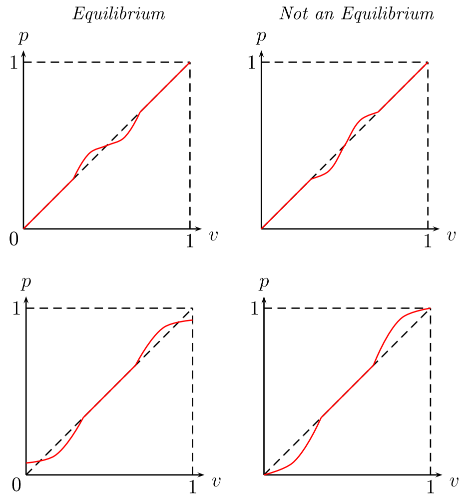


Fig. 1. Candidates for equilibrium prices. (For interpretation of the references to color, the reader is referred to the web version of this article.)

equilibria exhibiting monotone prices, i.e., prices that are (weakly) increasing in the value of the security.[3]

The main findings are threefold. First, despite restrictions on how much a trader can buy/sell in the market, when the presence of naive traders is not too high, sophisticated traders are able to arbitrage away any mispricing. However, partial mispricing (some values are mispriced but not others) or complete mispricing arise under a strong presence of naive traders. Second, even though mispricing can follow a complicated pattern, it exhibits a systematic feature: if there is overpricing at some values there must also be underpricing at higher values and, conversely, underpricing must be preceded by overpricing at some lower values. Typical pricing patterns under partial mispricing are illustrated in Fig. 1, which shows, for an asset taking values in [0, 1], candidates for equilibrium prices (left panel) and prices that could never arise in equilibrium (right panel).[4] Under complete mispricing, this feature translates into the well-known favorite-longshot bias (FLB) found in betting markets: bets with a low payout probability (longshots) are overpriced while bets with a high payout probability (favorites) are underpriced.[5] Finally, information about the order book, e.g. the bid-ask spread or the depth of the order book around transaction prices, can help an outside observer to discern whether the security is mispriced or not.

The basic intuition behind these results is the fact that, driven by her goal of maximizing expected monetary payoffs, a sophisticated trader may be *pivotal*, with her bid coinciding with the market price, *only* when prices are correct. This is because, given the auction mechanism considered here, each agent is constrained to name a single bid representing, respectively, the lowest acceptable price if the agent is a seller or the highest acceptable price if the agent is a buyer. Given this, when anticipating an increasing price schedule, a sophisticated trader will

[3] Note that whenever prices are *strictly* increasing they are informationally efficient given that they can be inverted to fully recover the actual value of the security.

[4] A mispricing interval occurs when prices (given by the red curve) are not on the diagonal.

[5] See for instance Snowberg and Wolfers [16] for a review of the evidence.

never choose a reservation price at which she expects mispricing to occur: if she is a seller and the price is above the expected value of the security she can increase profits by lowering her bid to make sure she trades at that price; if she is a buyer she can lower her bid to avoid buying an overpriced asset. A similar reasoning applies to instances of underpricing. Consequently, any mispricing must be driven by naive traders being pivotal. If their presence is sufficiently high, then due to limits on how much can be bought or sold in the market, sophisticated traders cannot be pivotal everywhere and mispricing ensues.

The described behavior of sophisticated traders ensures that regions of values where naive traders set prices typically exhibit overpricing of low values and underpricing of high values. This pattern is needed to preserve the incentives of sophisticated traders bidding right below or above a mispricing region.[6] To illustrate this, consider prices exhibiting the reverse pattern as in the top-right of Fig. 1, and assume that some sophisticated traders place their bids below prices in the mispricing region. In such a case, they would rather deviate and place their bids in the middle of the mispricing interval, i.e. at the value where prices go from under to overpricing. This is because buyers would like to buy an underpriced asset and thus would want to raise their reservation price or bid. Similarly, sellers wish to avoid selling when the asset is underpriced, and thus they would also want to raise their reservation price. Finally, differences in the density of bids can be exploited to infer the type of region observed prices belong to: a sparse order book around observed prices is indicative of mispricing since only naive traders bid at those prices, whereas both sophisticated and naive traders typically place bids at correct prices, leading to a thick order book. This is reminiscent of empirical studies finding that relatively big spreads in stock markets tend to be associated to mispricing (Sadka and Scherbina [15]).

From a design perspective, these results represent positive news for prediction markets, with a cautionary note to proponents of these markets. The good news is that the forecasting properties of prices do not critically depend on all traders being perfectly rational. However, what it is critical, as has been informally argued, is to attract enough smart money to the market (see [18, 4]). In addition, the results also suggest that disclosing information about the order book can help identify when prices yield biased forecasts.

This paper is related to the limits of arbitrage literature, which investigates whether mispricing can survive in equilibrium when the population includes both sophisticated and biased/naive traders. In these models mispricing exists because, facing uncertainty, risk aversion limits how much sophisticated traders invest in the market. Uncertainty may be about naive traders' beliefs (De Long et al. [1]) or about the amount of sophisticated traders in the market (Stein [17]). In contrast, uncertainty here stems from traders' private information about asset values, which makes it possible to analyze the ability of markets to aggregate information and produce unbiased forecasts. In terms of results, some papers find a tendency of *underreaction* to fundamentals. They also show the possibility that more arbitrage capacity does not necessarily translate into less mispricing: a stronger presence of sophisticated traders may increase price volatility, discouraging them from taking advantage of any expected mispricing. In contrast, I find that both under- and over-pricing coexist as long as there is mispricing and that, because of risk neutrality, increasing the presence of sophisticated traders mitigates mispricing.

The model is also associated to those dealing with the existence of fully revealing rational expectations equilibrium (REE) in which gains from trade are generated by noise traders (Hellwig [7], Kyle [11]), heterogeneous beliefs (Ottaviani and Sørensen [13]) or idiosyncratic

[6] It also arises when all traders in the market are naive.

preferences. Among the latter, Reny and Perry [14] is the closest to this paper, since it exhibits a two-sided auction with asymmetric information similar to the one studied here. However, their focus is on the existence of fully revealing REE in finite but large markets and not on the relationship between prices and fundamentals.

This paper is organized as follows. The model is laid out in Section 2. Section 3 characterizes equilibrium prices and pins down the different pricing regimes. Several extensions are discussed in Section 4 before the conclusion in Section 5.

## 2. The model

There is a unit mass of agents indexed in the unit interval $t \in [0, 1]$, which is endowed with the Lebesgue measure. A fraction $\gamma \in (0, 1)$ of them are sellers, each owning one unit of a security, with the remaining fraction being buyers, willing to buy at most one unit.[7] The value of the security $V \in [0, 1]$ is unknown with probability distribution $G(\cdot)$. Each agent receives a private signal $S \in [0, 1]$ stochastically related to $V$.[8] The mass of agents with signals below $s$ when the value of the security is $v$ is given by the probability distribution $F(s|v)$.

**Assumption 1.** $G$ is $C^2$ with density $g$ bounded away from 0 in $[0, 1]$. $F$ is $C^2$ with density $f$ bounded away from 0 in $[0, 1] \times [0, 1]$.

**Assumption 2.** $f(\cdot|\cdot)$ satisfies the strict monotone likelihood ratio property (MLRP).

The first assumption implies that the distribution of signals has full support for all values of the asset. That is, a trader receiving a signal $s \in [0, 1]$ cannot rule out any asset value in $[0, 1]$. The second assumption means that higher signals are more likely than lower signals when the asset value is high.

Buyers and sellers simultaneously submit reservation prices (*bids*) to buy and sell specifying, respectively, the maximum price willing to pay (buyers) and the minimum price willing to accept (sellers). Bids are restricted to be in $[0, 1]$. If the market price is $p$ and the value is $v$, a buyer bidding above $p$ gets a unit of the asset and a payoff of $v - p$ and a seller with a bid below $p$ trades her unit and receives a payoff $p - v$. If there is a positive mass of bids at $p$ there is the possibility of rationing, i.e., some traders bidding exactly $p$ may not trade. In this case, the traders bidding $p$ who end up with the object are chosen randomly.[9]

A fraction $\eta \leqslant 1$ of the trader population is naive, indexed in $[0, \eta]$, while traders in $(\eta, 1]$ are risk-neutral, sophisticated traders. The bidding behavior of naive traders is captured by the probability distribution $H(\cdot|v)$, where $\eta H(p|v)$ represents the mass of naive bids lower than or equal to $p$. As I explain below, by directly working with the distribution of naive bids rather than imposing behavioral constraints, the model includes as special cases some existing approaches to boundedly rational or biased behavior in finance and in behavioral game theory.[10] I assume that $H(\cdot|v)$ satisfies some regularity conditions, namely that it is differentiable, weakly monotonic

[7] I address in Section 4 the case in which the role of buyer/seller is endogenous.

[8] Capital letters denote random variables $(V, S)$ and lowercase letters denote realizations $(v, s)$.

[9] Reny and Perry [14] use the same tie-breaking rule.

[10] Given this modelling device and the pricing rule defined below, prices depend on the fraction of naive traders but not on how they are distributed across buyers and sellers. Thus, I do not make any assumptions on the proportion of naive traders that are sellers and on whether the distributions of naive buyers' and sellers' bids are identical.

with respect to values and has full support. Let $\underline{b}^H \geqslant 0$ be the lowest possible bid a naive trader may place and $\bar{b}^H \leqslant 1$ the highest naive bid, with $\underline{b}^H < \bar{b}^H$.

**Assumption 3.** $H(\cdot|v)$ has full support in $[\underline{b}^H, \bar{b}^H]$ for all $v \in [0,1]$ with a density bounded above and away from 0. $H(\cdot|\cdot)$ is $C^1$ in $(\underline{b}^H, \bar{b}^H) \times [0,1]$ and absolutely continuous in $[0,1] \times [0,1]$.

This assumption implies that the distribution of naive bids is atomless and strictly increasing in $(\underline{b}^H, \bar{b}^H)$ for all $v \in [0,1]$. It provides a clear contrast between the behavior of naive traders and the equilibrium behavior of sophisticated traders, since the latter may introduce atoms in their bid distribution when taking advantage of any mispricing introduced by the former. For instance, this assumption imposes the behavioral restriction that, generically, naive traders cannot use bidding strategies such as "trade at any price for all $s$" or "never trade for all $s$," which are implemented by choosing bids equal to zero or one, respectively.

The next assumption implies that $H(\cdot|v)$ first order stochastically dominates $H(\cdot|v')$ if $v > v'$. It is key to the existence of (weakly) monotone prices.

**Assumption 4.** $H(b|\cdot)$ is non-increasing in $[0,1]$ for all $b \in [0,1]$.

One can interpret this assumption, combined with the MLRP of signal distributions, as requiring that naive traders follow bidding strategies that are (weakly) increasing in their signals. Examples include noise or liquidity traders who bid randomly ($H(p|v) = p$ for all $v$), and traders bidding their interim valuations, $\mathbb{E}(V|s)$.[11] In addition, naive traders in some models of the limits of arbitrage (e.g. [17]) follow strategies consistent with this behavior.[12] The critical restriction is that naive behavior is not determined in equilibrium, given that $H$ does not depend on sophisticated traders' *equilibrium* strategies.[13]

Given a profile of measurable bidding strategies $\beta : [0,1] \times (\eta, 1] \to [0,1]$ with $\beta(s,t)$ denoting the bid of sophisticated trader $t$ when she receives signal $s$, let $B(p|V)$ be the mass of bids lower than or equal to $p$. Also, let $B_-(p|V)$ be the mass of bids strictly less than $p$. Accordingly,

$$B(p|v) := \eta H(p|v) + \int_{\eta}^{1} \int_{0}^{1} 1_{\{\beta(s,t) \leqslant p\}} f(s|v)\, ds\, dt, \tag{1}$$

and

$$B_-(p|v) := \eta H(p|v) + \int_{\eta}^{1} \int_{0}^{1} 1_{\{\beta(s,t) < p\}} f(s|v)\, ds\, dt, \tag{2}$$

11 In a continuum of agent economy, the latter represent *fully cursed* traders, who fail to account for the common value nature of the asset (Holt and Sherman [8], Kagel and Levin [10], Eyster and Rabin [2]), and agents who mistakenly believe that everybody shares their own information structure (Jehiel and Koessler [9]). These agents are featured in recent models of prediction markets (Manski [12], Gjerstad [5], Wolfers and Zitzewitz [19]).

12 Given the different market mechanisms the distributions of aggregate naive demand are not directly comparable. However, in the mentioned models individual demand is, as in here, an increasing function of fundamentals and it is independent of the behavior of arbitrageurs.

13 This does not mean that naive traders need not best respond given their beliefs, as is the case, for instance, with fully cursed traders.

where $1_{\{\cdot\}}$ is the indicator function. The market clearing price is given by the function $\rho : [0, 1] \to [0, 1]$ that satisfies

$$(1-\gamma) \in \big[B_-\big(\rho(v)\big|v\big), B\big(\rho(v)\big|v\big)\big] \quad \text{for all } v \in [0, 1]. \tag{3}$$

That is, if the price is $p$, the mass of bids above $p$, given by $1 - B(p|v)$, equals the mass of units for sale ($\gamma$) except, perhaps, when there is a positive mass of bids at $p$ in which case the rationing rule determines who of those bidding $p$ get one unit of the asset. This pricing rule guarantees that all buyers bidding strictly above $p$ get a unit of the security and all sellers bidding below $p$ sell their unit of the security.

Given this auction mechanism, the payoffs of a sophisticated seller and buyer at the market clearing price when they bid $b$ and receive signal $s$ are, respectively,

$$\pi^{sell}(s, b) := \mathbb{E}\big(\big(\rho(V) - V\big)\big(1 - \lambda(b, V)\big)1_{\{b \leqslant \rho(V)\}}\big|s\big), \tag{4}$$

and

$$\pi^{buy}(s, b) := \mathbb{E}\big(\big(V - \rho(V)\big)\lambda(b, V)1_{\{b \geqslant \rho(V)\}}\big|s\big), \tag{5}$$

where $\lambda(b, v)$ denotes the probability that a trader ends up with a unit after bidding $b$ when the value is $v$. Note that $\lambda(b, v) = 0$ if $b < \rho(v)$ and $\lambda(b, v) = 1$ if $b > \rho(v)$.

## 3. Results

In this section I investigate pricing patterns as a function of the fraction of naive traders ($\eta$) and their bidding behavior ($H$). To do so, I look for Bayes–Nash equilibria (BNE) in which sophisticated traders best respond to the equilibrium strategies of the other sophisticated traders, taking $H$ as given. In addition, I restrict attention to *monotone* equilibria, which exhibit prices $\rho(v)$ increasing in $v$. The two main results are stated in Propositions 1 and 2. The first characterizes equilibrium prices, whereas the second shows existence and uniqueness of monotone prices and identifies the different pricing regimes. All proofs are relegated to Appendix A. The exposition of results assumes that sophisticated traders are never rationed—Lemma 4 in Appendix A shows this is always the case in equilibrium.

### 3.1. Sophisticated bidding

First, I discuss the features of sophisticated traders' bidding behavior that drive the characterization of prices. To fix ideas, consider first the behavior of sophisticated seller $t$ receiving signal $s$ when she anticipates that prices in equilibrium are given by the increasing function $\rho(\cdot)$. Since there is a continuum of agents she cannot influence prices. Accordingly, her goal is to pick a reservation price $\beta(s, t) = b$ so as to maximize $\pi^{sell}(s, b)$, i.e. the expected difference $\rho(V) - V$, conditional on her private signal and on the fact that she sells her unit whenever $b \leqslant \rho(V)$.

Given this, there are two things she would never do, *regardless* of her signal, when facing a (strictly) increasing price function. First, she will never place a bid $b$ if $\rho(v) < v$ for all $v \in [v^b, v']$, where $v^b$ is the value at which $\rho(v^b) = b$ and $v' > v^b$. That is, if the security is underpriced at prices immediately above her bid $b$, seller $t$ would rather *raise* her bid to avoid making a loss by selling at those prices. Second, she will never place a bid $b$ if $\rho(v) > v$ for all $v \in [v'', v^b]$ with $v'' < v^b$. That is, if the security is overpriced at prices immediately below her bid $b$, she would *lower* her bid to make sure she sells at those prices. Note that if prices are constant in an interval of values the same reasoning applies by comparing the (constant) price to the expected value of the security in the interval.

It turns out that the same behavioral guidelines apply to sophisticated buyers, since buyers and sellers rank alternative bids similarly.

**Lemma 1.** *Buyers and sellers receiving the same signal* $s \in [0, 1]$ *have the same payoff ranking over bids.*

To see why this result holds, consider the relative ranking between two alternative bids $b$ and $b' < b$. The change in seller payoffs from bidding $b$ to bidding $b'$ is given by $\mathbb{E}((\rho(V) - V)1_{\{b' \leqslant \rho(V) \leqslant b\}}|s)$, which is just the negative of the change in buyer payoffs when switching from $b'$ to $b$. Thus, if seller payoffs go down when trading at prices in $[b', b]$ then buyer payoffs must increase by trading at prices in $[b', b]$. In such case, both the seller and the buyer prefer the higher bid $b$ over $b'$, since by picking $b$ the seller avoids trading while the buyer does trade at prices in $[b', b]$.[14,15]

As a direct consequence of the above behavioral rules, three things must be true in any monotone equilibrium:

(a) if a sophisticated trader places her bid $b$ inside the range of equilibrium prices, given by $[\rho(0), \rho(1)]$, there is *no mispricing* when the price is equal to $b$.
(b) If there is mispricing at prices just *above* a given sophisticated bid $b$, it must involve *overpricing* at prices close to $b$. Otherwise, the sophisticated trader that placed bid $b$ would rather deviate by increasing it, regardless of her signal.
(c) Similarly, if there is any mispricing at prices right *below* a given sophisticated bid $b$, there is *underpricing* at prices close to $b$.[16]

### 3.2. Equilibrium prices

Given facts (a)–(c), one can foresee a systematic price pattern arising in equilibrium. Because of (a), there is no mispricing ex post when sophisticated traders are pivotal, i.e. when the price equals the bid of some sophisticated trader. Mispricing could arise, however, at prices such that only naive traders are pivotal, in which case (b)–(c) typically lead to a pattern of *first-overpricing-then-underpricing*. I call such a pattern, a *local* favorite-longshot bias (FLB), since it is similar to the regular FLB except that it happens within each interval of values in which only naive traders are pivotal. In particular, if sophisticated traders are pivotal for some values but not for others, prices are typically characterized by a succession of intervals of values with and without mispricing. The following example illustrates the construction of equilibrium prices.[17]

[14] The only possible asymmetry between a buyer and a seller in a two-sided auction is driven by their attempt to affect prices in opposite directions, which is absent in a continuum economy.

[15] Lemma 1 also applies to one-sided auctions with a fixed supply of units and, as shown in [14], to economies in which agents' preferences include a private value component.

[16] As explained below, Assumption 3 guarantees that facts (b) and (c) also hold when either sophisticated trader bids below a mispricing area or above it, respectively.

[17] I am grateful to an anonymous referee for suggesting it.

**Example 1.** Consider a symmetric market ($\gamma = 0.5$) with half the traders being naive ($\eta = 0.5$). Let $V \sim U[0,1]$, and $H(p|v) = p$ for all $v \in [0,1]$, i.e., naive traders are pure noise traders. Finally, let $S \sim U[0,v]$, i.e. $F(s|v) = s/v$ if $s \in [0,v]$.[18]

To fix ideas, I focus on symmetric equilibria, that is, equilibria in which sophisticated traders use the same *monotone* bid function $\beta(s)$. This turn out to be without loss since all monotone equilibria yield the same prices and a symmetric equilibrium always exists. Let $\alpha(b)$ be the highest signal associated with a bid $b$ under $\beta$—when $\beta$ is strictly increasing, $\alpha$ is its inverse. If a sophisticated trader is pivotal at some price $p$ when the value is $v$ then the mass of sophisticated bids at or below $p$ is given by $(1-\eta)F(\alpha(p)|v)$. For $p$ to clear the market it must satisfy (1), i.e.

$$0.5 = 0.5H(p|v) + 0.5F\big(\alpha(p)\big|v\big),$$

which leads to

$$1 = p + \alpha(p)/v.$$

However, because of fact (a) we must have $p = v$, otherwise no sophisticated trader would bid $p$. That is,

$$1 = v + \frac{\alpha(v)}{v}, \tag{6}$$

which implies that $\alpha(v) = v - v^2$. If this function is increasing for all $v \in [0,1]$ then it defines a monotone bidding function $\beta$ that leads to perfect pricing. It is easy to see that $\alpha(v)$ is increasing for $v \leqslant 1/2$ and decreasing for $v > 1/2$. I argue that, in this case, equilibrium prices must exhibit mispricing for some values because $\alpha$ does not yield a well-defined bidding function. In particular, it is not possible to induce perfect pricing for some interval of values above $1/2$. The reason is that if there is perfect pricing at some $v > 1/2$ it is because all sophisticated traders with signals lower than $\alpha(v)$ are bidding below $v$. But then, because $\alpha$ is decreasing for $v > 1/2$, to get perfect pricing at some $v' > v$ we would need some of those traders to bid above $v'$, instead of below $v$. Hence, the region of values exhibiting mispricing should include the interval $[\underline{v}, 1]$ with $\underline{v} \leqslant 1/2$.

Accordingly, consider the case of no mispricing for values lower than $\underline{v}$ and mispricing for values above $\underline{v}$. The bidding function $\beta$ would then be determined by $\alpha(v)$ for bids in $[0, \underline{v}]$ and, because of fact (a), would jump above $\rho(1)$ for signals above $\alpha(\underline{v})$, e.g., by having $\beta(s) = 1$ for $s > \alpha(\underline{v})$. After inverting $\alpha$ we get

$$\beta(s) = \begin{cases} \frac{1-\sqrt{1-4s}}{2} & s < \alpha(\underline{v}), \\ 1 & s > \alpha(\underline{v}), \end{cases}$$

with the equilibrium price being

$$\rho(v) = \begin{cases} v & v < \underline{v}, \\ 1 - \frac{\alpha(\underline{v})}{v} & v \geqslant \underline{v}. \end{cases}$$

[18] The distribution of signals does not have full support and thus violates Assumption 1. Nonetheless, the main results continue to hold and the price pattern just described arises in equilibrium.

Given all this, to characterize both $\beta$ and $\rho$, we just need to pin down $\underline{v} \in [0, 1/2]$ such that (i) there is overpricing at values immediately above $\underline{v}$ and underpricing at values immediately below 1, and (ii) no sophisticated agent wants to deviate from $\beta(s)$. Overpricing happens whenever the mass of sophisticated traders bidding below $v$ is smaller than $0.5F(\alpha(v)|v)$, which is the mass of sophisticated bids yielding $\rho(v) = v$. Given the above argument, this can be achieved by having $\underline{v} < 1/2$, since $\alpha$ would be strictly increasing at such $\underline{v}$. This way the mass of sophisticated bids below any $v \in [\underline{v}, 1/2]$ would be $0.5F(\alpha(\underline{v})|v) < 0.5F(\alpha(v)|v)$. Similarly, underpricing will happen for values close to one as long as $\alpha(\underline{v}) > \alpha(1) = 0$, i.e., as long as $\underline{v} > 0$.

Regarding (ii), notice that, as I show in Appendix A (Fact 1), the expected difference between values and prices in $[\underline{v}, 1]$, given by $\mathbb{E}((\rho(V) - V)1_{\{V \in [\underline{v},1]\}}|s)$, is decreasing in $s$. This is because, by the MLRP of $F$, as $s$ goes up (underpriced) high values become more likely while (overpriced) low values become less likely. Accordingly, if a sophisticated seller with signal $\alpha(\underline{v})$ is indifferent between bidding below $\underline{v}$ or bidding 1, i.e., $\mathbb{E}((\rho(V) - V)1_{\{V \in [\underline{v},1]\}}|\alpha(\underline{v})) = 0$, then any seller with signal $s < \alpha(\underline{v})$ would be happy bidding $\beta(s) < \beta(\alpha(\underline{v})) = \underline{v}$ since she gets a positive expected payoff by trading when $v \in [\underline{v}, 1]$, and any seller with a higher signal is also happy by bidding one since she avoids a negative expected payoff by not trading when $v \in [\underline{v}, 1]$. Hence, by symmetry of preferences, no sophisticated trader would want to deviate from $\beta$. Using this zero expected payoff condition we solve for $\underline{v}$:

$$0 = \mathbb{E}\big(\big(\rho(V) - V\big)1_{\{V \in [\underline{v},1]\}}\big|\alpha(\underline{v})\big) = (2 - \underline{v})(1 - \underline{v}) - \log(\underline{v}).$$

The unique solution of this equation satisfying $\underline{v} < 1/2$ is given by $\underline{v} \simeq 0.316$ and $\alpha(\underline{v}) \simeq 0.216$. Fig. 2 shows the equilibrium bidding function and the corresponding equilibrium prices, which exhibit the local FLB in the mispricing interval.

Proposition 1 formally states that this pattern of alternating intervals of values with and without mispricing generalizes to all monotone equilibria. In particular, I show that there is a collection of intervals $[\underline{v}_k, \overline{v}_k]$, which may be empty (*perfect pricing*) or cover the whole interval $[0, 1]$ (*complete mispricing*), in which prices are determined by naive bids, whereas values outside those intervals are correctly priced. In any such interval, prices exhibit the local FLB, except in the unlikely case that naive bids lead to perfect pricing almost everywhere in the interval. These intervals and thus equilibrium prices are identified in a similar fashion as in the example: by simultaneously identifying each interval $[\underline{v}_k, \overline{v}_k]$ and the signal $s_k^*$ of the sophisticated trader indifferent about trading in such interval. In particular, $\underline{v}_k$, $\overline{v}_k$ and $s_k^*$ solve the following system of equations:

$$\rho(\underline{v}_k) \geqslant \underline{v}_k, \quad \text{with equality if } \underline{v}_k > 0; \tag{7}$$

$$\rho(\overline{v}_k) \leqslant \overline{v}_k, \quad \text{with equality if } \overline{v}_k < 1; \quad \text{and} \tag{8}$$

$$\mathbb{E}\big(\big(\rho(V) - V\big)1_{\{V \in (\underline{v}_k,\overline{v}_k)\}}\big|s\big) \geqslant 0\ (\leqslant 0) \quad \text{for all } s \leqslant s_k^*\ \big(s \geqslant s_k^*\big). \tag{9}$$

Conditions (7)–(8) imply perfect pricing at the boundaries of a mispricing interval except, maybe, when the interval starts at zero in which case overpricing is possible, or when the interval ends at one, in which case underpricing may happen, as it is the case in the above example. Condition (7) ensures that a trader with signal $s < s_k^*$ bids below $\rho(\underline{v}_k)$ whereas a trader with $s > s_k^*$ bids above $\rho(\overline{v}_k)$. Hence, the mass of sophisticated bids below market prices in $[\underline{v}_k, \overline{v}_k]$ is equal to $(1 - \eta)F(s_k^*|v)$.

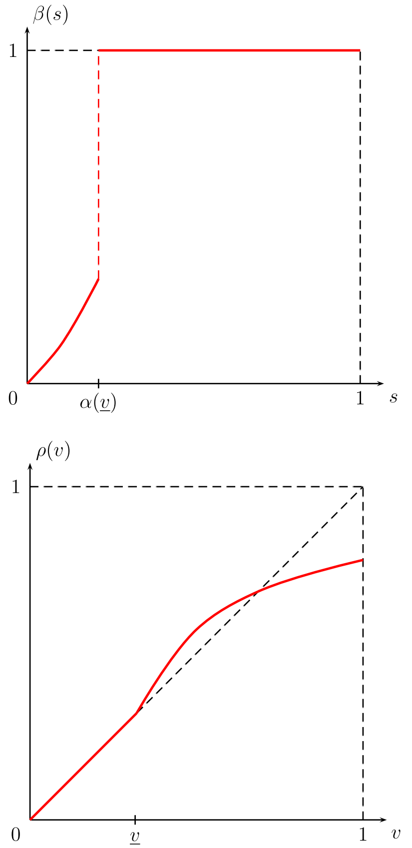


Fig. 2. Equilibrium bidding strategy and prices in Example 1.

**Proposition 1** *(Equilibrium prices). Let Assumptions 1–4 be satisfied. Prices in any monotone equilibrium are characterized by a set* $\mathcal{V} = \bigcup_{k=1}^{K} [\underline{v}_k, \overline{v}_k]$ *with* $\overline{v}_k \leqslant \underline{v}_{k+1}$ *for all* $k = 1, \ldots, K$ *and a collection of signals* $\{s_k^*\}$ *with* $s_k^* < s_{k+1}^*$ *satisfying (7)–(9) such that*

(i) $\rho(v) = v$ *for all* $v \notin \mathcal{V}$; *and*
(ii) *if* $v \in [\underline{v}_k, \overline{v}_k]$ *then* $\rho(v)$ *is given by*

$$1 - \gamma = \eta H\big(\rho(v)\big|v\big) + (1 - \eta) F\big(s_k^*\big|v\big). \tag{10}$$

This proposition also implies that the mass of sophisticated bids below prices in $(\overline{v}_k, \underline{v}_{k+1})$ satisfies the market clearing condition associated to perfect pricing, given by $1 - \gamma = B(v|v)$. That is, if $\beta$ represents equilibrium bidding strategies then

$$\int_{\eta}^{1} \int_{0}^{1} 1_{\{\beta(s,t) \leqslant v\}} f(s|v)\, ds\, dt = 1 - \gamma - \eta H(v|v), \quad \text{for all } v \in [\overline{v}_k, \underline{v}_{k+1}]. \tag{11}$$

The next corollary formally states that the local FLB arises in $[\underline{v}_k, \overline{v}_k]$. In this context, the full support of $F$ and $G$ (Assumption 1) and the MLRP (Assumption 2) guarantee that (9) is satisfied.

The continuity of $H$ (Assumption 3) ensures that there is no rationing of sophisticated traders and, in addition, that the local FLB holds when $\underline{v}_1 = 0$ or $\overline{v}_K = 1$. Finally, Assumption 4 rules out the possibility of prices being constant in an interval of values, except under very special circumstances.[19]

**Corollary 1.** *If $\mathcal{V}$ is non-empty, in any given $[\underline{v}_k, \overline{v}_k]$ there exist $v'_k$, $v''_k$ satisfying $\underline{v}_k < v'_k \leqslant v''_k < \overline{v}_k$ such that $\rho(v) > v$ in $(\underline{v}_k, v'_k)$, and $\rho(v) < v$ in $(v''_k, \overline{v}_k)$.*

It is worth emphasizing that Corollary 1 allows for the possibility that some values in a mispricing interval are correctly priced. This happens when $H$ satisfies $1 - \gamma = \eta H(v|v) + (1 - \eta)F(s^*_k|v)$ in $(v_1, v_2) \subset [\underline{v}_k, \overline{v}_k]$. Therefore, when I refer to "complete mispricing" I actually mean that naive traders set prices for all asset values. Nonetheless, these instances of naive traders setting prices right are unlikely in the sense that any small shift of $H$ would eliminate them.

### 3.3. *Pricing regimes*

The next result states that monotone equilibria exist and that monotone equilibrium prices are essentially unique. In addition, it sheds light on how the presence of naive traders affects prices: there is a positive lower bound on the fraction of naive bidders below which there is perfect pricing; and there is an upper bound above which prices are always set by naive bidders, with sophisticated traders relegated to bidding outside the price range.

**Proposition 2** *(Existence of equilibrium). Let Assumptions 1–4 be satisfied. A monotone equilibrium exists for all $\eta \in [0, 1]$ and all such equilibria exhibit essentially the same prices. Furthermore, there exists $\underline{\eta} \in (0, \min\{\gamma, 1 - \gamma\})$ such that $\mathcal{V}$ is the empty set for all $\eta \leqslant \underline{\eta}$, and $\overline{\eta} \leqslant 1$ such that $\mathcal{V} = [0, 1]$ for all $\eta > \overline{\eta}$.*

Before focusing on the different pricing regimes associated to different values of $\eta$, I briefly explain the intuition behind existence and uniqueness. Following a similar logic as in the example above, the proof of existence essentially shows that continuity of $F$ and $H$ and the MLRP guarantee the existence of a collection of triplets $\{(s^*_k, \underline{v}_k, \overline{v}_k)\}_{k=1}^K$ satisfying (7)–(9). Given these triplets, we can always find a function $\alpha$ mapping bids to signals such that the mass of sophisticated bids below $b$ is given by $(1 - \eta)F(\alpha(b)|v)$ for all $b \in [0, 1]$, and exhibits the following properties: it is constant for $b \in [\underline{v}_k, \overline{v}_k]$ and, is given in $[\overline{v}_k, \underline{v}_{k+1}]$ by

$$1 - \gamma = \eta H(v|v) + (1 - \eta)F\bigl(\alpha(v)\big|v\bigr).$$

That is, $\alpha$ leads to no sophisticated bidding in $[\underline{v}_k, \overline{v}_k]$ and to perfect pricing in $[\overline{v}_k, \underline{v}_{k+1}]$ for all $k$. Notice that, for $\alpha(\cdot)$ to be induced by well-defined bidding functions $\beta(\cdot, t)$, it has to be (weakly) increasing. Otherwise, $\alpha(b) < \alpha(b')$ for some $b' < b$ would mean that fewer traders place bids below $b$ than below $b' < b$, a contradiction. The MLRP and $H$ being weakly decreasing in $v$ (Assumption 4) guarantee that $\alpha$ is increasing in $[\overline{v}_k, \underline{v}_{k+1}]$ and, in addition, that market clearing prices are monotone. Given this, a symmetric equilibrium always exists since the mass of bids $(1 - \eta)F(\alpha(b)|v)$ can be implemented by a symmetric bidding function, as illustrated in Example 1.

[19] See Lemma 4 in Appendix A.

Uniqueness of monotone prices is based on the fact that, due to the strict MLRP, each triplet $(s_k^*, \underline{v}_k, \bar{v}_k)$ satisfying (7)–(9) and Corollary 1 is unique. To see why, consider again Example 1, in which equilibrium prices are pinned down by $\underline{v}_1 = \underline{v}$, $\bar{v}_1 = 1$ and $s_1^* = \alpha(\underline{v})$. If we lower $\underline{v}_1$ by making some sophisticated traders with signals immediately below $s_1^*$ bid above $\underline{v}_1$, the mass of bids below $\underline{v}_1$ goes down, leading to more overpricing and less underpricing in $[\underline{v}_1, 1]$. But then, sellers with signals close to $s_1^*$ would have an incentive to bid below $\underline{v}_1$, implying that some traders would bid suboptimally in equilibrium, a contradiction. Using similar arguments one can show that increasing $\underline{v}_1$ or decreasing $\underline{v}_2$ would also lead to some sophisticated traders behaving suboptimally. The proof uses this logic to show uniqueness in the presence of multiple mispricing intervals and provides an algorithm to characterize $\{(s_k^*, \underline{v}_k, \bar{v}_k)\}_{k=1}^K$.

The last part of Proposition 2 establishes the existence of three pricing regimes depending on the proportion of naive traders: perfect pricing, partial mispricing (i.e. with both mispricing and perfect pricing intervals) and complete mispricing. In order to provide some intuition, consider the setup of Example 1 but let $\eta$ now be arbitrary. The market clearing condition (6) associated to perfect pricing becomes

$$0.5 = \eta v + (1-\eta)\frac{\alpha(v)}{v},$$

which leads to

$$\alpha(v) = \frac{0.5}{1-\eta} v - \frac{\eta}{1-\eta} v^2.$$

First consider the perfect pricing regime. This requires $\alpha$ to be well-defined ($0 \leqslant \alpha(v) \leqslant v$) and increasing in $[0, 1]$[20]; otherwise $\beta$ would not be well-defined. The former condition is true for all $v \in [0, 1]$ whenever $\eta \leqslant 0.5$ while the latter is satisfied when $\eta \leqslant 0.25$. Therefore, we are in the perfect pricing regime for all $\eta \leqslant \underline{\eta} = 0.25$, with $\beta$ being the inverse of $\alpha$.

Next, notice that $\alpha$ is not well-defined or decreasing for all values in the complete mispricing regime. Given that $\alpha(v) > v$ for $v < (\eta - 0.5)/\eta$ and $\alpha$ is decreasing whenever $v > 0.25/\eta$, $\alpha$ is either not well-defined or decreasing when $(\eta - 0.5)/\eta \geqslant 0.25/\eta$. Thus, the complete mispricing regime includes all $\eta \geqslant \bar{\eta} = 0.75$. In this case, there exists a signal $s_1^*$ satisfying $\mathbb{E}(\rho(V) - V|s_1^*) = 0$ such that $\beta(s) \leqslant \rho(0)$ if $s \leqslant s_1^*$ and $\beta(s) \geqslant \rho(1)$ if $s > s_1^*$. Prices are pinned down by plugging $s_1^*$ into market clearing condition (10), which yields

$$\rho(v) = \begin{cases} \frac{0.5}{\eta} - \frac{1-\eta}{\eta} & v \leqslant s_1^*, \\ \frac{0.5}{\eta} - \frac{1-\eta}{\eta}\frac{s_1^*}{v} & v > s_1^*. \end{cases}$$

Fig. 3 shows the symmetric bidding function and equilibrium prices for $\eta = 0.75$, which exhibit the FLB globally. Finally, for $\eta$ between $\underline{\eta}$ and $\bar{\eta}$ there are intervals of values in which $\alpha$ is increasing and intervals in which it is either not well-defined or decreasing. This leads to the partial mispricing regime, as Example 1 illustrates.

*The role of Assumption 3*

Before discussing potential extensions of the model I introduce the following result, which sheds light on the link between mispricing patterns and Assumption 3.

[20] Recall that signals lie in $[0, v]$.

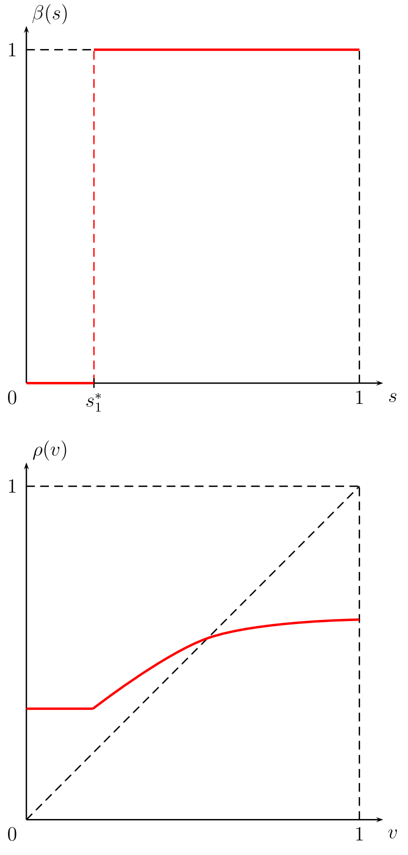


Fig. 3. Equilibrium bidding strategy and prices in Example 1 when $\eta = 0.75$.

**Corollary 2.** *Let Assumptions 1, 2 and 4 be satisfied.*

(i) *If Assumption 3 holds then, for all $\eta \geqslant \bar{\eta}$, $\rho(v) > v$ for all $v$ sufficiently close to zero and $\rho(v) < v$ for all $v$ sufficiently close to one*; *and*
(ii) *there is a distribution of naive bids for all $\eta \in [0, 1]$ such that $\rho(v) = v$ for all $v \in [0, 1]$. This distribution violates Assumption 3 for all $\eta > \min\{\gamma, 1 - \gamma\}$.*

Part (i) is a restatement of Corollary 1 when $\mathcal{V} = [0, 1]$, i.e., the FLB holds *globally* when the fraction of naive traders is sufficiently high. Also, notice that making $H$ equal to the distribution of sophisticated bids associated with $\eta = 0$ leads to perfect pricing for all $\eta$, given that $\eta$ drops from the market clearing condition. But this implies that $H$ would exhibit atoms, thus violating Assumption 3. To see why, recall that to have correct pricing at $v$, (11) requires the mass of sophisticated bids below $v$ to be $1 - \gamma - \eta H(v|v)$. If $H$ is atomless then $H(0|0) = 0$, so to have $\rho(0) = 0$, there needs to be a mass $1 - \gamma$ of sophisticated bids at 0. This is only possible if $\eta \leqslant \gamma$. Likewise, to set $\rho(1) = 1$ there needs to be a mass $\gamma$ of bids at 1, which can only happen if $\eta < 1 - \gamma$. Thus, a necessary condition for the absence of mispricing when $H$ is atomless is that $\eta \leqslant \min\{\gamma, 1 - \gamma\}$.

## 4. Extensions

I briefly discuss the following modifications of the model and their effect on equilibrium prices: (i) the possibility of endogeneizing the buyer/seller roles; (ii) allowing sophisticated traders to condition their bids on prices (full demand schedules); and (iii) the introduction of aggregate uncertainty.

*Endogenous roles*

In most asset markets agents decide whether to buy or to sell, rather than being exogenously assigned to one side of the market. A way to introduce this choice is to endow all traders with a unit of the asset and let a trader be a seller when her bid falls below the market price and be a buyer otherwise. It turns out that prices in this modified double auction constitute a special case of the original model. Notice that sophisticated traders face the same incentives as before due to the symmetry of preferences: if a trader is happy with being a buyer for prices below $b$, she is also happy with being a seller for prices above $b$. Hence, her bidding behavior does not change compared to the case of exogenous roles. What changes is that now she always trades, so the mass of units traded is equal to one half. Thus, the market clearing price coincides with the price in a market with exogenous roles and $\gamma = 0.5$.

*Unit demand/supply schedules*

Allowing sophisticated traders to submit (unit) demand schedules rather than single bids reduces the extent of mispricing. Such demand schedules let each agent to condition her decision to trade one unit on the realized price, rather than submitting a single reservation price. Equilibrium prices in this environment also show a partitional structure with different pricing regimes, although none of them exhibits complete mispricing (except at $\eta = 1$). To get an idea of how prices arise in this market, assume that sophisticated traders can condition their decision to trade on the realized price while naive traders still submit a single bid and are symmetrically distributed among buyers and sellers (i.e., the mass of naive sellers is $\eta\gamma$).[21] In this context, if the price is $\rho(v)$, a sophisticated seller would agree to sell her unit if $\rho(v) \geqslant v$, and a sophisticated buyer would only buy if $\rho(v) \leqslant v$. Given this behavior, equilibrium prices are pinned down by (i) finding the market clearing conditions associated to the three possible scenarios: overpricing, underpricing and correct pricing; and (ii) identifying which one is satisfied at each $v$.

If $\rho(v) > v$, aggregate supply equals $\eta\gamma H(\rho(v)|v) + (1-\eta)\gamma$ and demand is $\eta(1-\gamma)(1-H(\rho(v)|v))$, which yields the market clearing condition

$$H\big(\rho(v)\big|v\big) = 1 - \frac{\gamma}{\eta}. \tag{12}$$

If $\rho(v) < v$, supply is $\eta\gamma H(\rho(v)|v)$ and demand is $\eta(1-\gamma)(1-H(\rho(v)|v)) + (1-\eta)(1-\gamma)$. Thus, $\rho(v)$ must satisfy

$$H\big(\rho(v)\big|v\big) = \frac{1-\gamma}{\eta}. \tag{13}$$

[21] The rationale for this type of market is the following. If the cost of submitting an offer is zero and there are no restrictions on short sales, sophisticated traders could replicate any demand schedule by simultaneously submitting multiple bids and asks. On the other hand, naive traders would lack the expertise to create complex bidding schemes.

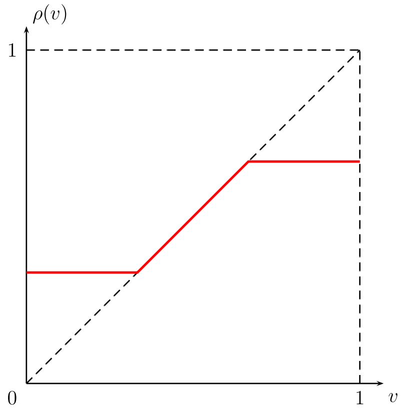


Fig. 4. Equilibrium prices with demand schedules when $\gamma = 0.5$ and $\eta = 0.75$.

Finally, if $\rho(v) = v$, sophisticated traders are indifferent between trading or not. In this case, supply lies between $\eta\gamma H(v|v)$ and $\eta\gamma H(v|v) + (1-\eta)\gamma$ and demand must fall between $\eta(1-\gamma)(1-H(v|v))$ and $\eta(1-\gamma)(1-H(v|v)) + (1-\eta)(1-\gamma)$.

Notice that prices satisfying (12) and (13) are (weakly) increasing, given that their RHS are constant in $v$ and $H(b|\cdot)$ is decreasing for all $b$. In addition, prices given by (12) are lower than those satisfying (13), since $\frac{1-\gamma}{\eta} > 1 - \frac{\gamma}{\eta}$. Thus, if (12) leads to overpricing at $v$ then (13) cannot yield underpricing at $v$ and vice versa, i.e., at most one equation is consistent. In addition, it could be that (12) or (13) are not satisfied because its RHS lies outside $[0, 1]$. When both equations are inconsistent or cannot be satisfied there must be perfect pricing at $v$. This is because the inconsistency of (12) stems from the fact that demand under overpricing is too weak, relative to supply, to yield prices above values. Likewise, demand is too strong to sustain underpricing when (13) is inconsistent. Hence, the only possibility left is to adjust sophisticated demand and supply so that market clearing yields $\rho(v) = v$. Given this, there are three distinct pricing regimes:

(i) *Perfect pricing*: For $\eta \leqslant \min\{\gamma, 1-\gamma\}$ prices equal values for all $v$, given that (12)–(13) cannot be satisfied. In the context of Example 1, allowing for demand schedules increases $\eta$ from 0.25 to 0.5.

(ii) *One-way mispricing*: For $\eta \in (\min\{\gamma, 1-\gamma\}, \max\{\gamma, 1-\gamma\}]$ either (12) or (13) is never satisfied. Hence, mispricing involves either underpricing or overpricing, respectively. In the underpricing case, there must be perfect pricing at values close to zero, since (13) is never consistent at $v = 0$ ($H(0|0) = 0 < \frac{1-\gamma}{\eta}$), and underpricing at values close to one, since $H(1|1) = 1 > \frac{1-\gamma}{\eta}$. In the overpricing case, there is overpricing at values close to zero and correct pricing at values close to one—(12) is not consistent at high enough $v$.

(iii) *Global FLB*: For $\eta > \max\{\gamma, 1-\gamma\}$ there exist $0 < v' < v'' < 1$ such that prices are above values for $v < v'$ and below values for $v > v''$, given that (12) implies $\rho(0) > 0$ and (13) implies $\rho(1) < 1$. Fig. 4 shows equilibrium prices when $\gamma = 0.5$, $\eta = 0.75$, when $H(p|v) = p$ for all $v$. In this example, (12) yields $\rho(v) = 1/3$, which is consistent for $v < 1/3$, and (13) yields $\rho(v) = 2/3$, which is consistent for $v > 2/3$. Accordingly, $\rho(v) = v$ in $[1/3, 2/3]$.

*Prices when the fraction of naive traders is unknown*

An important characteristic of the model is that, unlike in some limits-of-arbitrage models (e.g. [17]) and in noisy-REE models (see [7]), there is no uncertainty about the demand of naive traders, except for the uncertainty caused by not observing $v$. This implies that, except in special cases, prices are fully revealing. A natural way to introduce aggregate uncertainty is to make the fraction of naive traders a random variable whose distribution is known by sophisticated traders, as in [17]. In an online appendix at http://dx.doi.org/10.1016/j.jet.2012.05.020, I provide an example for which the results stated above continue to hold *in expectation*. Specifically, there are intervals in the price range where sophisticated traders avoid placing their bids, which are associated with expected mispricing, i.e. $\mathbb{E}(V|\rho(V,\widetilde{\eta})=p)\neq p$ ($\widetilde{\eta}$ denotes the random fraction of naive traders). Additionally, any such interval starts with (expected) overpricing ($\mathbb{E}(V|\rho(V,\widetilde{\eta})=p)<p$) and ends in underpricing.

## 5. Conclusion

The analysis presented here yields several insights that may be helpful in the design of prediction markets and in better assessing their forecasting performance. The first is that limiting the size of trades, as is frequently done in these markets, may lead to mispricing by limiting the arbitrage function of sophisticated traders. The second is that information from the order book may be useful to outside observers since it helps identify instances of mispricing. Finally, price forecasts exhibit a tendency to be biased upward for low values and biased downward for high values whenever there is mispricing.

## Appendix A

### *A.1. Proofs of Proposition 1 and Corollary 1*

The proofs of Proposition 1 and Corollary 1 hinge upon a series of lemmas formalizing the intuition about sophisticated bidding presented above. In addition to Lemma 1, I show that no sophisticated trader would place bids below a mispricing interval that starts with prices below values or above one that ends with overpricing (Lemma 2). Lemma 3 states that sophisticated traders avoid placing bids in mispricing intervals. Finally, I show that no rationing of sophisticated traders takes place in equilibrium, since any atom is solely created by naive traders and can only happen in very special cases (Lemma 4).

**Proof of Lemma 1.** Let $\rho(V)$ be the price function resulting from strategy profile $\beta(\cdot,\cdot)$, and assume buyer $t$ and seller $t'$ bid $b$ when they receive signal $s$, i.e. $\beta(s,t)=\beta(s,t')=b$. If we subtract (4) from (5) we get

$$\pi^{buy}(s,t)=\pi^{sell}\big(s,t'\big)+\mathbb{E}\big(\big(V-\rho(V)\big)\big|s\big). \tag{14}$$

Since the last term does not depend on $b$, a buyer and a seller receiving the same signal will have the same preference ranking over bids. □

Let $\rho_{+}^{-1}(b):=\max\{v\colon\ \rho(v)=b\}$, $\rho_{-}^{-1}(b):=\min\{v\colon\ \rho(v)=b\}$.

**Lemma 2** *(Sophisticated bidding (I)). Let $\beta$ be sophisticated traders' strategy profile in a monotone equilibrium and $(v_1,v_2)$ be a non-degenerate set of values.*

(i) *If* $\rho(v) < v$ *a.e. in* $(v_1, v_2)$ *and there is a trader* $t$ *with* $\beta(s,t) < \rho(v_1)$ *for some* $s$*, then there exists* $v' \in (\rho_-^{-1}(\beta(s,t)), v_1]$ *such that* $\rho(v) \geqslant v$ *for all* $v \in (\rho_-^{-1}(\beta(s,t)), v']$*, with strict inequality in a non-null subset.*

(ii) *If* $\rho(v) > v$ *a.e. in* $(v_1, v_2)$ *and there is a trader* $t$ *with* $\beta(s,t) > \rho(v_2)$ *for some* $s$*, then there exists* $v' \in [v_2, \rho_+^{-1}(\beta(s,t)))$ *such that* $\rho(v) \leqslant v$ *for all* $v \in (v', \rho_+^{-1}(\beta(s,t)))$*, with strict inequality in a non-null subset.*

**Proof.** To prove (i) assume that $\rho(v) \leqslant v$ holds for all $v \in (\rho_-^{-1}(\beta(s,t)), v_1)$. Then, given that $\mathbb{E}((V - \rho(V))1_{\{\rho(V)<v_2\}}|s) > 0$ for all $s$, a buyer would strictly prefer to bid $v_2$ than $\beta(s,t)$. Since preferences are symmetric, a seller would also prefer to bid $v_2$. A symmetric argument applies to (ii). □

The following fact is used in the proofs of Lemma 3 and Proposition 1.

**Fact 1.** *Let Assumptions 1 and 2 be satisfied. If* $\rho(v) > v$ *a.e. in* $(v_1, v_2)$ *and* $\rho(v) < v$ *a.e. in* $(v_2, v_3)$ *with* $\rho(\cdot)$ *increasing, then for all* $s \in (0,1)$*:*

(i) *If* $\mathbb{E}((V - \rho(V))1_{\{V\in[v_1,v_3]\}}|s) \leqslant 0$*, then* $\mathbb{E}((V - \rho(V))1_{\{V\in[v_1,v_3]\}}|s') < 0$ *for all* $s' < s$*;*

(ii) *If* $\mathbb{E}((V - \rho(V))1_{\{V\in[v_1,v_3]\}}|s) \geqslant 0$*, then* $\mathbb{E}((V - \rho(V))1_{\{V\in[v_1,v_3]\}}|s') > 0$ *for all* $s' > s$*.*

**Proof.** Let $\mathbb{E}((V - \rho(V))1_{\{V\in[v_1,v_3]\}}|s) \leqslant 0$. Thus,

$$\frac{1}{f(s)}\int_{v_1}^{v_2}\big(\rho(V) - V\big) f(s|v)g(v)\,dv \geqslant \frac{1}{f(s)}\int_{v_2}^{v_3}\big(V - \rho(V)\big) f(s|v)g(v)\,dv. \tag{15}$$

The strict MLRP of $F$ (Assumption 2) implies that $\frac{f(s'|v')}{f(s|v')} > \frac{f(s'|v)}{f(s|v)}$ for all $s' < s$ and all $v' \in [v_1, v_2)$ and $v \in [v_2, v_3)$. Given this, (15) yields

$$\int_{v_1}^{v_2}\big(\rho(V) - V\big) f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)\,dv > \int_{v_2}^{v_3}\big(V - \rho(V)\big) f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)\,dv. \tag{16}$$

But since $f(s') > 0$ for all $s'$ by the full support of $F$ and $G$ (Assumption 1), (16) implies that $\mathbb{E}((\rho(V) - V)1_{\{V\in[v_1,v_2]\}}|s') > \mathbb{E}((V - \rho(V))1_{\{V\in[v_2,v_3]\}}|s')$. A symmetric argument applies to part (ii). □

**Lemma 3** *(Sophisticated bidding (II)). If Assumptions 1–3 hold the mass of sophisticated traders submitting bids in* $\{\rho(v)\colon \rho(v) \neq v\}$ *is zero in a monotone equilibrium, except perhaps when there is a positive mass at* $\rho(0)$ *or at* $\rho(1)$*, and* $1 - \gamma = B(\rho(0)|v)$ *for all* $v \in [0, \rho_+^{-1}(0)]$ *(complete rationing) or* $1 - \gamma = B_-(\rho(1)|v)$ *for all* $v \in [\rho_-^{-1}(1), 1]$ *(no rationing), respectively.*

**Proof.** First, notice that if the set $\{v\colon \rho(v) \neq v\}$ is non-null, it can be represented by a countable union of non-degenerate disjoint intervals. This is because, for any $v$ such that $\rho(v) > v$, the monotonicity of $\rho$ implies that $\rho(v') > v'$ for all $v' \in [v, \rho(v))$. Similarly, for any $v$ such that $\rho(v) < v$, we have that $\rho(v') < v'$ for all $v' \in (\rho(v), v]$. That is, every $v \in \{v\colon \rho(v) \neq v\}$ belongs to a non-degenerate interval of values mispriced in the same direction. The union of disjoint intervals must be at most countable, otherwise $\{v\colon \rho(v) \neq v\}$ would not have finite measure.

By Lemma 1, I only need to look at a buyer's incentives. The proof is divided in two cases, depending on whether the price is strictly increasing in $v$ or it is constant in an interval of values.

*Case 1*: $\rho(\cdot)$ *is strictly increasing in* $[0,1]$. I need to show that no sophisticated buyer is best-responding by bidding in the interior of a price interval in which $\rho(v) \neq v$. Assume otherwise that a buyer bids in an interval $(\rho(v_1), \rho(v_2))$ where $v > \rho(v)$. In this case, she prefers to bid $v_2$ to any bid $b \in (\rho(v_1), \rho(v_2))$, given that her payoff increases by $\mathbb{E}((V - \rho(V))1_{\{\rho(V)\in(b,\rho(v_2))\}}|s)$, which is strictly positive for all $s$. If, on the other hand, $v < \rho(v)$ in $(v_1, v_2)$, a buyer would prefer to bid below $\rho(v_1)$ given that $\mathbb{E}((V - \rho(V))1_{\{\rho(V)\in(\rho(v_1),b)\}}|s) < 0$ for all $s$.

*Case 2*: $\rho(\cdot)$ *is constant in an interval of values*. I first show that if $\rho(\cdot)$ is constant in an interval of values (i.e. the distribution of prices has an atom) a sophisticated buyer will only bid at the atom if she gets the object with probability zero or one, depending on whether the expected value (conditional on her signal) of $\rho(V) - V$ at the atom is positive or negative, respectively. Otherwise, she would bid slightly above or below to either avoid trading or being rationed. I next show that these conditions cannot be satisfied at an atom in the interior of the price range. Therefore, the only possibility left for a sophisticated buyer bidding in $\{\rho(v)\colon v - \rho(v) \neq 0\}$ is to bid at the boundaries, with the condition that she does not trade almost surely when she bids $\rho(0)$ and that she trades with probability one when bidding $\rho(1)$.

To prove the first part of the argument, assume there is an atom at $b \in (0,1)$, associated to values in $(v_1, v_2)$. If there is a mass of sophisticated bids at $b$, a buyer with signal $s$ might bid $b$ under one of these scenarios: (i) $\mathbb{E}((V-b)1_{\{\rho(V)=b\}}|s) = 0$; (ii) $\mathbb{E}((V-b)1_{\{\rho(V)=b\}}|s) > 0$ with $\lambda(b,v) = 1$ for all $v \in (v_1, v_2)$ (no rationing); (iii) $\mathbb{E}((V-b)1_{\{\rho(V)=b\}}|s) < 0$ with $\lambda(b,v) = 0$ for all $v \in (v_1, v_2)$ (complete rationing).

In case (i), she is indifferent between bidding slightly above or below $b$. However, Fact 1 implies that there can be at most one signal satisfying (i).[22] Therefore the mass of bids at $b$ due to (i) is zero. $\lambda(b,v) = 1$ in (ii), otherwise she would bid above $b$ to get the object with probability one. Finally, in (iii) she may bid at $b$ only if the probability of getting the object is zero ($\lambda(b,v) = 0$). Since in each of the latter two cases $\lambda(b,\cdot)$ is required to be zero or one in the whole interval $(v_1, v_2)$, there cannot be two traders bidding at $b$ with distinct signals satisfying (ii) and (iii), respectively. Accordingly, either (ii) or (iii) holds for all the sophisticated bidders bidding $b$.

Now I show that for (ii) and (iii) to hold we need $\rho(1) = b$ and $\rho(0) = 0$, respectively. Assume (ii) is satisfied for all bidders bidding $b$ and let $\underline{s}$ be the lowest signal associated to $b$. Accordingly, a trader receiving $\underline{s}$ bids optimally at $b$ if

$$\mathbb{E}\big(\big(V - \rho(V)\big)1_{\{p\leqslant b\}}\big|\underline{s}\big) \geqslant 0, \tag{17}$$

and

$$\mathbb{E}\big(\big(V - \rho(V)\big)1_{\{p> b\}}\big|\underline{s}\big) \leqslant 0. \tag{18}$$

By Lemma 2, we can apply Fact 1 to (17) and (18).[23] Hence, all sophisticated traders with signals above $\underline{s}$ will bid at or above $b$ (assuming $\lambda(b,v) = 1$). Likewise, given (18) and the fact that $\mathbb{E}((V-b)1_{\{\rho(V)=b\}}|\underline{s}) \geqslant 0$, all sophisticated traders with $s < \underline{s}$ will bid strictly below $b$.

[22] For (i) to hold, $v < b$ in the lower part of $(v_1, v_2)$ and $v > b$ in the upper part of $(v_1, v_2)$.

[23] By the linearity of expectations, the conclusions of Fact 1 also apply to a succession of intervals satisfying the conditions in the lemma.

For $\lambda(b,v)=1$ we need the mass of sellers bidding strictly less than $b$ be equal to the mass of buyers bidding at $b$ or above. That is, for all $v\in(v_1,v_2)$,

$$\gamma\big[\eta H(b|v)+(1-\eta)F(\underline{s}|v)\big]=(1-\gamma)\big[\eta\big(1-H(b|v)\big)+(1-\eta)\big(1-F(\underline{s}|v)\big)\big]. \tag{19}$$

Given that $B_-(b|v)=\eta H(b|v)+(1-\eta)F(\underline{s}|v)$, the above expression is satisfied when $B_-(b|v)=1-\gamma$ for all $v\in(v_1,v_2)$.

Now assume that $b<\rho(1)$, i.e. $v_2<1$ and $\rho(v)>b$ for all $v>v_2$. For that to happen we need $B(b|v)<1-\gamma$ for all $v>v_2$. But this implies, by the continuity of $F$ and $H$ (Assumptions 1 and 3), that there exists $v'<v_2$ such that $B_-(b|v)<B(b|v)\leqslant 1-\gamma$ for all $v\geqslant v'$, which contradicts that $\lambda(b,v)=1$ for all $v\in(v_1,v_2)$. Hence, (ii) is only possible in equilibrium if $b=\rho(1)$ and $v_2=1$.

A symmetric argument applies when (iii) is satisfied for almost all sophisticated traders bidding at $b$.

Finally, if $\rho(1)$ is not an atom, the probability of rationing is zero and a buyer bidding $\rho(1)$ always trades. In this case there can be a positive mass of sophisticated bids at $\rho(1)<1$, since any buyer bidding $\rho(1)<1$ is indifferent between any two bids in $[\rho(1),1]$. A symmetric argument can be made for bids at $\rho(0)>0$. □

Lemma 3 allows for the possibility of having sophisticated bids placed at an atom, at $\rho(0)$ or at $\rho(1)$, of the price distribution if either sellers or buyers bidding at the atom trade with probability one, respectively. However, as the next lemma shows, atoms can only occur for very particular naive share and bid distributions.

**Lemma 4** *(No atoms). Let Assumptions 1–4 hold. In any monotone equilibrium if there exists $v_1<v_2$ such that $\rho(v)=b$ on $(v_1,v_2)$ then*

(a) $\mathbb{E}((V-\rho(v))1_{\{V<v_2\}}|s)\geqslant 0$ *for all $s$, and $H(\rho(v)|v)=\frac{1-\gamma}{\eta}$ for all $v\leqslant v_2$;*
(b) $\mathbb{E}((V-\rho(v))1_{\{V<v_2\}}|s)\leqslant 0$ *for all $s$, and $H(\rho(v)|v)=\frac{\eta-\gamma}{\eta}$ for all $v\geqslant v_1$.*

Lemma 4 says that atoms in the distribution of $\rho(V)$ are created by naive traders, and that very special circumstances are needed for atoms to occur: the share of naive bids must be very high compared to $\gamma$ (or to $1-\gamma$); naive traders must determine prices at the low (high) end of the price range, with those prices being low (high) enough so that they do not encourage sophisticated traders to bid below (above) the atom; and the distribution of naive bids must be independent of asset values in the interval of values associated with the atom.[24]

**Proof.** Assume there is an interval $(v_1,v_2)$ such that $\rho(v)=b$ for all $v\in(v_1,v_2)$. By Lemma 2 and Fact 1, if there exists a trader with signal $s$ bidding below (above) $b$ then it is optimal for all traders with signals below (above) $s$ to also bid below (above) $b$. Accordingly, let $\underline{s}\in[0,1]$ be the highest signal associated with bids below $b$, and $\bar{s}\geqslant\underline{s}$ the lowest signal associated with bids above $b$. This implies that

[24] As an example of equilibrium prices being constant let $H(b|v)=b$, i.e., naive traders bid uniformly in $[0,1]$. In this case, $\rho(v)=\frac{1-\gamma}{\eta}$ for all $v$ can be sustained by having all sophisticated traders bidding above $\frac{1-\gamma}{\eta}$. This can be achieved as long as $\eta$ is high enough so that $\mathbb{E}(V|0)\geqslant\frac{1-\gamma}{\eta}$.

$$B_-(b|v) = \eta H(b|v) + (1-\eta)F(\underline{s}|v),$$

and

$$B(b|v) = \eta H(b|v) + (1-\eta)F(\bar{s}|v).$$

There are two possible cases, depending on whether a positive mass of sophisticated bids is placed at $b$ or not, i.e. whether $\underline{s} < \bar{s}$ or $\underline{s} = \bar{s}$.

If there is no positive mass of sophisticated bids at $b$, we have that $B(b|v) = B_-(b|v) = 1-\gamma$ for all $v \in (v_1, v_2)$. Since $F(s|v)$ is strictly decreasing in $v$ for all $s \in (0,1)$ and $H(b|v)$ is non-increasing in $v$ for all $b \in [0,1]$ by Assumption 4, $B_-(b|v) = 1-\gamma$ for all $v \in (v_1, v_2)$ only if $\underline{s} = 0$ or $\underline{s} = 1$.

a) $\underline{s} = 0$: in this case $H(b|v) = \frac{1-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. But then, we need $\mathbb{E}((V - \rho(V))1_{\{\rho(V) \leqslant b\}}|s) = \mathbb{E}((V - \rho(V))1_{\{V < v_2\}}|s) \geqslant 0$ for all $s$, otherwise some sophisticated traders would rather bid below $b$. Finally, prices below $b$ are completely determined by naive bids, since no sophisticated trader bids below $b$, i.e. $H(\rho(v)|v) = \frac{1-\gamma}{\eta}$ for all $v \leqslant v_1$.
b) $\underline{s} = 1$: in this case $H(b|v) = \frac{\eta-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. In addition, we need $\mathbb{E}((V - \rho(V))1_{\{V < v_2\}}|s) \leqslant 0$ for all $s$. Since no sophisticated trader bids above $b$, prices above $b$ are given by $H(\rho(v)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \geqslant v_2$.

If there is a positive mass of sophisticated bids at $b$, Lemma 3 applies, requiring either that $B_-(b|v) = 1-\gamma$ or $B(b|v) = 1-\gamma$. The former requires $\underline{s} = 0$ or $\underline{s} = 1$, while the latter can be possible only if $\bar{s} = 0$ or $\bar{s} = 1$. Therefore, they reduce to the same conditions on $H(\cdot|\cdot)$ and $\mathbb{E}((V - \rho(V))1_{\{\rho(V) \leqslant b\}}|s)$. □

**Proof of Proposition 1.** The first step of the proof is to define $\mathcal{V}$. As shown in the proof of Lemma 3, the monotonicity of $\rho$ implies that $\{v\colon\ \rho(v) \neq v\}$ is a collection of non-degenerate disjoint intervals whenever it has positive measure. To construct $\mathcal{V}$, first augment this collection by adding any interval $(v_1, v_2)$ with the following three features: $\rho(v) = v$ in $(v_1, v_2)$; $v_1$ and $v_2$ are the boundaries of adjacent intervals in $\{v\colon\ \rho(v) \neq v\}$; and no element of $[v_1, v_2]$ is in the range of $\beta$, the equilibrium profile of sophisticated bidding strategies. That is, starting from the collection of intervals of mispriced values, we add to it any interval with correct prices that connects two intervals of mispriced values as long as no sophisticated trader bids in it. Given this augmented collection, define $\mathcal{V}$ to be its closure. Notice that these two steps involve merging in a single interval adjacent intervals in which only naive traders are pivotal. Accordingly, $\mathcal{V}$ can be represented as a (at most countable) union of disjoint closed intervals $[\underline{v}_k, \bar{v}_k]$. Furthermore, by construction, any such interval must exhibit mispricing at values immediately above $\underline{v}_k$ and below $\bar{v}_k$. Assume $\mathcal{V}$ is non-empty, otherwise Proposition 1 holds trivially. In what follows, I focus on seller behavior by Lemma 1.

Denote $B^*(b|v)$ the mass of sophisticated bids at or below $b$ given $v$. Consider first the case of $B^*(\cdot|\cdot)$ being atomless. Accordingly, by Lemma 3, we must have $B^*(\rho(v)|v) = B^*(\rho(\underline{v}_k)|v)$ for all $v \in [\underline{v}_k, \bar{v}_k]$, where $\rho(v)$ is given by

$$1-\gamma = \eta H\bigl(\rho(v)\big|v\bigr) + B^*\bigl(\rho(\underline{v}_k)\big|v\bigr). \tag{20}$$

We need to show that $B^*(\rho(\underline{v}_k)|v) = F(s_k^*|v)$, for all $k$ and all $v \in [\underline{v}_k, \bar{v}_k]$, with $s_k^*$ satisfying (7)–(9).

Before doing so I show that, when $B^*$ is atomless, the pricing pattern in $[\underline{v}_k, \overline{v}_k]$ is the one described in Corollary 1. First, we must have $\rho(0) > 0$ and $\rho(1) < 1$ since a mass of at least $1-\gamma$ is needed at zero to set $\rho(0) = 0$, which is not possible since both $H$ and $B^*$ are atomless. Likewise, $\rho(1) = 1$ would require a mass of bids at one of at least $\gamma$. Accordingly, for each interval $[\underline{v}_k, v'_k]$, there exist $v'_k$, $v''_k$ with $\underline{v}_k < v'_k \leqslant v''_k < \overline{v}_k$ such that $\rho(v) > v$ in $(\underline{v}_k, v'_k]$, and $\rho(v) < v$ in $[v''_k, \overline{v}_k)$.[25] This is because either $\underline{v}_k = 0$ and thus $\rho(v) > v$ in $[0, \rho(0))$, or $\underline{v}_k > 0$, in which case some sophisticated bidders bid below $\rho(\underline{v}_k)$ and part (a) of Lemma 2 applies. A symmetric reasoning applies to the presence of underpricing in $[v''_k, \overline{v}_k]$.

Given this, for a seller with signal $s$ to bid $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ it must be that

$$\mathbb{E}\big(\big(V-\rho(V)\big)1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}\big|s\big) + \sum_{k'>k}\mathbb{E}\big(\big(V-\rho(V)\big)1_{\{V\in(\underline{v}'_k,\overline{v}'_k)\}}\big|s\big) \leqslant 0$$

with $\mathbb{E}((V-\rho(V))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) \leqslant 0$, otherwise she would bid above $\rho(\overline{v}_k)$. But, given the pricing pattern in $[\underline{v}_k, \overline{v}_k]$, these inequalities hold strictly for all $s' < s$ by Fact 1. Hence, bidding above $\rho(\overline{v}_k)$ is strictly dominated by bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ for all sellers with $s' < s$. A symmetric argument can be used for all $s' > s$ when a seller with signal $s$ finds optimal to bid in $[\rho(\overline{v}_k), \underline{v}_{k+1}]$. Therefore, $B^*(\rho(\underline{v}_k)|v) = (1-\eta)F(s^*_k|v)$ for some signal $s^*_k > 0$. Moreover, by the continuity of $H$ and $F$, $s^*_k$ needs to satisfy (7) if $\underline{v}_k > 0$ and (8) whenever $\overline{v}_k < 1$, given that $\rho(v) = v$ in $(\overline{v}_{k-1}, \underline{v}_k)$ and in $(\overline{v}_k, \underline{v}_{k+1})$. Finally, condition (9) is the equilibrium condition for a seller with $s \leqslant s^*_k$ ($s > s^*_k$) to optimally bid below $\rho(\underline{v}_k)$ (above $\rho(\overline{v}_k)$), which implies that $s^*_{k-1} < s^*_k$ for all $k > 1$.

Now assume that $B^*$ has an atom. Since $H$ is atomless, $B^*$ cannot have an atom in $(\rho(0), \rho(1))$, since it would induce prices to be constant in some interval of values. This would lead to a mispricing interval where sophisticated bids are placed, a contradiction of Lemma 3. Therefore, $B^*$ can have an atom only in $\{\rho(0), \rho(1)\}$.

If $B^*$ has an atom at $\rho(0)$, the price distribution may have an atom at $\rho(0)$. In such case, prices are equal to $\rho(0)$ for values close to $v = 0$ and, by Lemma 4, we must have $\rho(0) > 0$, $B^*(\rho(0)|v) = 1-\eta$ for all $v$ and $H(\rho(0)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \leqslant \rho_+^{-1}(\rho(0))$. Accordingly, $s^*_1 = 1$ and (10) is satisfied, which also implies that $\rho(1) < 1$ since $H$ is atomless. If the price distribution does not have an atom at $\rho(0)$, $\rho(0)$ is given by (20), i.e.

$$1-\gamma = \eta H\big(\rho(0)\big|0\big) + B^*\big(\rho(0)\big|0\big). \tag{21}$$

Hence, if a mispricing interval starts at $\rho(0)$ (i.e. $\underline{v}_1 = 0$), Lemma 2 applies to the interval $[0, \overline{v}_1]$ and, by Fact 1, there exists a signal $s^*_1 > 0$ satisfying (9) such that $B^*(\rho(0)|v) = (1-\eta)F(s^*_1|v)$. In either case, prices in $[0, \overline{v}_1]$ satisfy Corollary 1.

Finally, let $B^*$ have an atom at $\rho(1)$. If the price distribution has an atom at $\rho(1)$, Lemma 4 implies that all sophisticated traders bid at or above $\rho(1) < 1$ for all $v$ and $H(\rho(1)|v) = \frac{1-\gamma}{\eta}$ for all $v \geqslant \rho_-^{-1}(\rho(1))$. Thus, $s^*_1 = 0$ and (10) is satisfied, which also implies that $\rho(0) > 0$. If the price distribution does not have an atom at $\rho(1)$, $\rho(1)$ must be given by

$$1-\gamma = \eta H\big(\rho(1)\big|1\big) + B^*_-\big(\rho(1)\big|1\big), \tag{22}$$

where $B^*_-(b|v)$ is the mass of sophisticated bids strictly lower than $b$ when the value is $v$. Therefore, if a mispricing interval ends at $\rho(1)$, Lemma 2 applies to the interval $[\underline{v}_K, 1]$ and, by Fact 1,

[25] In what follows, I use the convention, $\overline{v}_0 = 0$ and $\underline{v}_{K+1} = 1$.

there exists a signal $s_K^* < 1$ satisfying (9) such that $B^*(\rho(\underline{v}_K)|v) = (1-\eta)F(s_K^*|v)$. Finally, prices in $[\underline{v}_K, 1]$ also satisfy Corollary 1. □

*A.2. Proof of Proposition 2*

Let $\mathcal{H}(v) := H(v|v)$. As stated above, Proposition 1 implies that the mass of sophisticated bids can be represented by $B(b|v) = (1-\eta)F(\xi(b)|v)$, where $\xi(\cdot)$ is an increasing function mapping bids to signals with the following properties: it is constant in mispricing intervals and, in intervals with correct prices, it is equal to the quantile function $\alpha(\cdot,\cdot)$ implicitly defined by

$$1-\gamma = \eta\mathcal{H}(v) + (1-\eta)F\big(\alpha(v,\eta)\big|v\big). \tag{23}$$

That is, $\alpha(v,\eta)$ represents the highest signal associated to sophisticated bids below $v$ such that $\rho(v) = v$ when the fraction of naive traders is $\eta$.[26] Both the uniqueness and the pricing regimes parts of Proposition 2 heavily rely on the properties of $\alpha$.

The next series of lemmas, leading to the proof of Proposition 2, show that $\alpha$ is increasing for small $\eta$; non-monotonic for intermediate levels of $\eta$; and either not well-defined or decreasing for all $v$ when $\eta$ is high enough. Prices must equal values everywhere in the first case, and there must be mispricing at values for which $\alpha$ is either not defined or decreasing.

In what follows, $D_i$ is the partial derivative with respect to the $i$th argument.

**Lemma 5.** *If Assumptions 1–4 are satisfied the following statements are true*:

- (i) *$\alpha(0,\eta)$ is well-defined for $\eta < \gamma$, strictly positive and increasing in $\eta$; $\alpha(1,\eta)$ is well-defined for $\eta < 1-\gamma$, strictly less than one and decreasing in $\eta$.*
- (ii) *If $D_1\alpha(v,\eta) \leqslant 0$ then $D_1\alpha(v,\eta') < 0$ for all $\eta' > \eta$ for which $\alpha(v,\eta)$ is well-defined.*
- (iii) *There exists $\underline{\eta} \in (0, \min\{\gamma, 1-\gamma\})$ such that $\alpha(\cdot,\eta)$ is well-defined and non-decreasing for all $\eta \leqslant \underline{\eta}$, and it is non-monotonic or decreasing for all $\eta > \underline{\eta}$ in the subset of values where it is well-defined.*
- (iv) *If $\mathcal{H}'(v) > 0$ for all $v$ such that $\mathcal{H}(v) = 1-\gamma$, there exists $\overline{\eta} \in [\underline{\eta}, 1)$ such that $\alpha(\cdot,\eta)$ is decreasing whenever it is well-defined for all $\eta > \overline{\eta}$.*

**Proof.** Part (i): since $\mathcal{H}(0) = 0$, $\alpha(0,\eta) = F^{-1}(\frac{1-\gamma}{1-\eta}|0)$, which is well-defined if $\eta < \gamma$. Since $F(\cdot|v)$ has full support for all $v$ and $\frac{1-\gamma}{1-\eta}$ is increasing in $\eta$, $\alpha(0,\eta)$ is increasing in $\eta$. Similarly, $\mathcal{H}(1) = 1$ so $\alpha(1,\eta) = F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|1)$ is well-defined for $\eta < 1-\gamma$ and decreasing in $\eta$.

Part (ii): by the a.e. differentiability of $H$ and $F$ (Assumptions 1 and 3), $\alpha(v,\eta)$ is a.e. differentiable. I differentiate both sides of (23) to obtain $D_1\alpha(v,\eta)$:

$$D_1\alpha(v,\eta) = -\frac{\frac{\eta}{1-\eta}\mathcal{H}'(v) + D_2F(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{24}$$

Note that $f(\cdot|\cdot) > 0$ by the full support assumption and $D_2F(\cdot|\cdot) < 0$ by strict MLRP. Therefore, for $D_1\alpha(v,\eta) \leqslant 0$ we need the numerator of (24) to be non-positive, i.e.

$$\frac{\eta}{1-\eta}\mathcal{H}'(v) + D_2F\big(\alpha(v,\eta)\big|v\big) \geqslant 0. \tag{25}$$

[26] Expression (23) is the analogue of the market clearing condition in the limit economy of [14] for the pure common values case with naive traders.

Thus, if we show that (25) implies

$$\frac{\partial}{\partial\eta}\left[\frac{\eta}{1-\eta}\mathcal{H}'(v)+D_2F\big(\alpha(v,\eta)\big|v\big)\right]>0,$$

which is equivalent to

$$\frac{\mathcal{H}'(v)}{(1-\eta)^2}>-D_2f\big(\alpha(v,\eta)\big|v\big)D_2\alpha(v,\eta),\tag{26}$$

then we would have shown that if the numerator of (24) is negative, it becomes more negative as $\eta$ grows. This will suffice to prove part (ii) of the lemma.

Given that $D_2\alpha(v,\eta)=\frac{1-\gamma-\mathcal{H}(v)}{(1-\eta)^2f(\alpha(v,\eta)|v)}$, (26) can be expressed as

$$\mathcal{H}'(v)>-\big(1-\gamma-\mathcal{H}(v)\big)\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.\tag{27}$$

Therefore, we need to prove that (25) implies (27). There are two possible cases: $\mathcal{H}(v)<1-\gamma$ and $\mathcal{H}(v)>1-\gamma$.[27] But before considering them, notice that the strict MLRP of $f$ implies that $\frac{D_2f(s|v)}{f(s|v)}\in[\frac{D_2F(s|v)}{F(s|v)},\frac{-D_2F(s|v)}{1-F(s|v)}]$ for all $s\in(0,1)$ and all $v$.[28] These bounds come from the fact that $\frac{F(s|v)}{f(s|v)}$ is decreasing in $v$ and $\frac{1-F(s|v)}{f(s|v)}$ is increasing in $v$ for all $s$, i.e.

$$\frac{\partial}{\partial v}\left[\frac{F(s|v)}{f(s|v)}\right]=\frac{f(s|v)D_2F(s|v)-D_2f(s|v)F(s|v)}{f^2(s|v)}\leqslant 0,\tag{28}$$

and

$$\frac{\partial}{\partial v}\left[\frac{1-F(s|v)}{f(s|v)}\right]=\frac{-f(s|v)D_2F(s|v)-D_2f(s|v)(1-F(s|v))}{f^2(s|v)}\geqslant 0.\tag{29}$$

*Case 1*: $\mathcal{H}(v)<1-\gamma$. If we divide both sides of (25) by $F(\alpha(v,\eta)|v)$,[29] we obtain

$$\frac{\eta}{1-\eta}\frac{\mathcal{H}'(v)}{F(\alpha(v,\eta)|v)}\geqslant-\frac{D_2F(\alpha(v,\eta)|v)}{F(\alpha(v,\eta)|v)}\geqslant-\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v)=\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}$ in the above expression and multiplying both sides by $1-\gamma-\mathcal{H}(v)$ we get

$$\mathcal{H}'(v)\frac{\eta(1-\gamma-\mathcal{H}(v))}{1-\gamma-\eta\mathcal{H}(v)}\geqslant-\big(1-\gamma-\mathcal{H}(v)\big)\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.\tag{30}$$

Since $\mathcal{H}(v)<1-\gamma$, $\gamma\in(0,1)$ and $\eta\in(0,1)$,[30] $\frac{\eta(1-\gamma-\mathcal{H}(v))}{1-\gamma-\eta\mathcal{H}(v)}$ is strictly positive and less than one. Hence, (30) implies (27) given that $\mathcal{H}'(v)>0$ by (25).

*Case 2*: $\mathcal{H}(v)>1-\gamma$. Two subcases need to be considered. If $D_2f(\alpha(v,\eta)|v)\leqslant 0$ the right-hand side of (27) is non-positive. Thus, (27) is satisfied for all $v$ such that $\mathcal{H}'(v)>0$ and all $\eta$. When $D_2f(\alpha(v,\eta)|v)>0$, dividing both sides of (25) by $1-F(\alpha(v,\eta)|v)$ leads to[31]

27 If $\mathcal{H}(v)=1-\gamma$, (27) is satisfied given that $\mathcal{H}'(v)>0$ is needed for (25) to hold.

28 These bounds are well-defined since $F(s|v)\in(0,1)$ for all $s\in(0,1)$ by Assumption 1.

29 $F(\alpha(v,\eta)|v)>0$ whenever $\mathcal{H}(v)<1-\gamma$: if $F(\alpha(v,\eta)|v)=0$, then $1-\gamma=\eta\mathcal{H}(v)\leqslant\mathcal{H}(v)$.

30 If (25) holds then $\eta>0$. Also, $\alpha(\cdot,1)$ is not well-defined.

31 $F(\alpha(v,\eta)|v)<1$ whenever $\mathcal{H}(v)>1-\gamma$: if $F(\alpha(v,\eta)|v)=1$ then $1-\gamma=\eta\mathcal{H}(v)+1-\eta\geqslant\mathcal{H}(v)$.

$$\frac{\eta}{1-\eta}\frac{\mathcal{H}'(v)}{1-F(\alpha(v,\eta)|v)} \geqslant -\frac{D_2F(\alpha(v,\eta)|v)}{1-F(\alpha(v,\eta)|v)} \geqslant \frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v)$ and rearranging terms, the above inequality becomes

$$\begin{aligned}\eta\mathcal{H}'(v) &\geqslant \big(1-\eta+\eta\mathcal{H}(v)-(1-\gamma)\big)\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}\\ &\geqslant -\big(1-\gamma-\mathcal{H}(v)\big)\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.\end{aligned} \tag{31}$$

The last inequality implies (27) and holds because $1-\eta+\eta\mathcal{H}(v) \geqslant \mathcal{H}(v)$.

Part (iii): Before proving this part, note that $\alpha(\cdot,\eta)$ is well-defined in $[0,1]$ iff $\eta \leqslant \eta_\gamma := \min\{\gamma, 1-\gamma\}$, given that $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta} \in [\frac{1-\gamma-\eta}{1-\eta}, \frac{1-\gamma}{1-\eta}]$ for all $v \in [0,1]$.

First, I show that there exists $\underline{\eta}_l > 0$ such that $\alpha(\cdot,\eta)$ is non-decreasing in $[0,1]$ for all $\eta < \underline{\eta}_l$. Since $F(s|v)$ is increasing in $s$ and decreasing in $v$, we have that, for $\eta \leqslant \eta_\gamma$ and all $v \in [0,1]$, $\alpha(v,\eta) \in [F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|0), F^{-1}(\frac{1-\gamma}{1-\eta}|1)] \subset (0,1)$. This implies that $D_2F(\alpha(v,\eta)|v) < 0$ for all $v \in (0,1)$. If $v \leqslant \underline{b}^H$ then $\alpha(v,\eta) = F^{-1}(\frac{1-\gamma}{1-\eta}|v)$ which is strictly increasing in $v$ for $\eta \leqslant \eta_\gamma$. Similarly, when $v \geqslant \bar{b}^H$ we have that $\alpha(v,\eta) = F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|v)$ is strictly increasing. Finally, notice that

$$D_1\alpha(v,\eta) \to \frac{-D_2F(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)} > 0 \quad \text{as } \eta \to 0 \text{ for all } v \in \big(\underline{b}^H, \bar{b}^H\big).$$

By the continuity of $\alpha$, $D_2F$ and $f$ (Assumption 1), we can find $\underline{\eta}_l > 0$ such that $\alpha(v,\eta)$ is non-decreasing for all $\eta \leqslant \underline{\eta}_l$.

Next, I show that there exists $\underline{\eta}_u < \eta_\gamma$ such that $\alpha(v,\eta)$ is strictly decreasing for some $v$ for all $\eta > \underline{\eta}_u$. By the continuity of $\alpha$, the existence of $\underline{\eta}_u$ is implied by the fact that, for all $\eta \geqslant \eta_\gamma$, there is a non-null subset of values in which $\alpha(\cdot,\eta)$ is strictly decreasing:

1. If $\eta \geqslant \gamma$, then $\frac{1-\gamma}{1-\eta} \geqslant 1$. Since $\frac{1-\gamma-\eta}{1-\eta} < 1$ and $\mathcal{H}$ is continuous, there exists an interval of values $[\underline{v}, \bar{v}]$ such that $\frac{1-\gamma-\eta\mathcal{H}(\underline{v})}{1-\eta} = 1$ and $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}$ is strictly decreasing in $v$ for all $v \in [\underline{v}, \bar{v}]$. Hence, by the full support assumption, $\alpha(\underline{v},\eta) = 1$ and $\alpha(v,\eta) < 1$ for all $v \in (\underline{v}, \bar{v}]$, which implies that $D_1\alpha(v,\eta) < 0$ for some subset of $(\underline{v}, \bar{v}]$.
2. If $\eta \geqslant 1-\gamma$, we have that $\frac{1-\gamma-\eta}{1-\eta} \leqslant 0$. Since $\frac{1-\gamma}{1-\eta} > 0$, there exists an interval of values $[\underline{v}', \bar{v}']$ such that $\frac{1-\gamma-\eta\mathcal{H}(\bar{v}')}{1-\eta} = 0$ and $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}$ is strictly decreasing in $v$ for all $v \in [\underline{v}', \bar{v}']$. Hence, $\alpha(\bar{v}',\eta) = 0$ and $\alpha(v,\eta) > 0$ for all $v \in [\underline{v}', \bar{v}')$, which implies that $D_1\alpha(v,\eta) < 0$ or some subset of $[\underline{v}', \bar{v}')$.

Finally, it remains to be shown that $\underline{\eta}_l = \underline{\eta}_u = \underline{\eta}$. By part (ii) of the lemma, if $D_1\alpha(v;\eta) \leqslant 0$ then $D_1\alpha(v;\eta') < 0$ for all $\eta' \in (\eta, \bar{\eta}_\gamma)$. But this also implies that if $D_1\alpha(v,\eta) \geqslant 0$, then $D_1\alpha(v,\eta'') > 0$ for all $\eta'' < \eta$. Therefore, given that $D_1\alpha(v;\cdot)$ is well-defined and continuous in $[0,\eta_\gamma]$ for all $v$, $\underline{\eta}_l = \underline{\eta}_u = \underline{\eta}$ and is given by the highest $\eta$ such that $D_1\alpha(v,\eta) \geqslant 0$ for all $v$, i.e.

$$\underline{\eta} := \sup_{\eta}\left\{\eta \in (0,1)\colon\ \eta \leqslant -\frac{D_2F(F^{-1}(\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}|v)|v)}{H'(v) - D_2F(F^{-1}(\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}|v)|v)}\ \forall v\right\}. \tag{32}$$

Part (iv): note that, as $\eta \to 1$, $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\gamma} \to \infty$ for all $v$ such that $\mathcal{H}(v) < 1-\gamma$ and $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\gamma} \to -\infty$ for all $v$ such that $\mathcal{H}(v) > 1-\gamma$. Thus, for high enough $\eta$, $\alpha(\cdot,\eta)$ is only well-defined in a small neighborhood of all $v$ such that $\mathcal{H}(v) = 1-\gamma$. If for any such $v$ $\mathcal{H}'(v) > 0$, then $\alpha(\cdot, \eta)$ will be decreasing in such neighborhood.[32] By part (ii) of the lemma, if $\alpha(\cdot, \eta)$ is decreasing, it is decreasing for all $\eta' > \eta$. □

**Fact 2.** *Let $\mathcal{S}$ be a measurable subset of $[0, 1]$ and $s \in (0, 1)$ be such that $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for some $v \in [0, 1)$. Then, $D_2\mathbb{P}(\mathcal{S}|v) \geqslant D_2F(s|v)$.*

**Proof.** Assume $\mathcal{S} \cap [s, 1]$ is a non-null set, otherwise $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for all $v$ by the full support of $F(\cdot|v)$ for all $v$. Since $\mathbb{P}(\mathcal{S}|v) = F(s|v) = \mathbb{P}([0, s]|v)$, we have that

$$\mathbb{P}\big([s, 1] \cap \mathcal{S}|v\big) = \mathbb{P}\big([0, s] \setminus \mathcal{S}|v\big).$$

By the strict MLRP of $F(\cdot|\cdot)$, the left-hand side is strictly greater than the right-hand side for all $v' > v$. Thus, $D_2[\mathbb{P}(\mathcal{S}|v) - \mathbb{P}([0, s]|v)] \geqslant 0$. □

**Lemma 6.** *If $\alpha(\cdot, \eta)$ is strictly decreasing in some interval $[v_1, v_2]$ then any monotone equilibrium price $\rho(\cdot)$ satisfies $\rho(v) \neq v$ a.e. in $[v_1, v_2]$.*

**Proof.** Assume $\rho(v) = v$ and $\rho(v') = v'$ for some $v' > v$ with $\alpha(v, \eta) > \alpha(v', \eta)$. Accordingly, if the mass of sophisticated bids below $v$ is given by bidders with signals in $[0, \alpha(v, \eta)]$, then the mass of bids below $v' > v$ is strictly smaller than the mass of bids below $v$, a contradiction. Hence, it must be that there is an alternative, well-defined function $B^a(\cdot|\cdot)$ determining the mass of sophisticated bids such that $B^a(v|v) = 1 - \gamma - \eta\mathcal{H}(v)$ $(= (1 - \eta)F(\alpha(v, \eta)|v))$ for all $v \in [v_1, v_2]$. Since $\alpha(\cdot, \eta)$ is decreasing in that interval, to get correct prices (25) requires that

$$\frac{d}{dv}B^a(v|v) = -\eta\mathcal{H}'(v) < (1 - \eta)D_2F\big(\alpha(v, \eta)\big|v\big).$$

Denoting $\beta^a$ the alternative profile of sophisticated bidding functions, we have that

$$B^a(v|v) = \int_\eta^1 \int_0^1 1_{\{\beta^a(s,t) \leqslant v\}} f(s|v)\, ds\, dt = \int_\eta^1 \mathbb{P}\big(\mathcal{S}^a(v, t)\big|v\big)\, dt,$$

where $\mathcal{S}^a(v, t) = \{s \in [0, 1]\colon\ \beta^a(s, t) \leqslant v\}$.

By Fact 2, $D_2\mathbb{P}(\mathcal{S}^a(v, t)|v) \geqslant D_2F(s^a(v, t)|v)$ with $s^a(v, t)$ being the signal such that $\mathbb{P}(\mathcal{S}^a(v, t)|v) = F(s^a(v, t)|v)$. Accordingly, given that $D_1B^a(v|v) \geqslant 0$,

$$\frac{d}{dv}B^a(v|v) = D_1B^a(v|v) + D_2B^a(v|v) \geqslant \int_\eta^1 D_2F\big(s^a(v, t)\big|v\big)\, dt.$$

Given this, in order to prove that there is no $B^a(\cdot|\cdot)$ leading to prices that equal values in $[v_1, v_2]$, it is enough to show that $\int_\eta^1 D_2F(s^a(v, t)|v)\, dt \geqslant (1 - \eta)D_2F(\alpha(v, \eta)|v)$ whenever $\int_\eta^1 F(s^a(v, t)|v)\, dt = (1 - \eta)F(\alpha(v, \eta)|v)$.

[32] In this case there is a unique $v$ such that $\mathcal{H}(v) = 1 - \gamma$. Assume otherwise that there are two such values $v$, $v'$ such that $\mathcal{H}'(v)$, $H'(v') > 0$. Since $\mathcal{H}(0) = 0$ and $\mathcal{H}(1) = 1$, by the continuity of $\mathcal{H}(\cdot)$, there would be a value $v'' \in$ such that $\mathcal{H}(v'') = 1 - \gamma$ and $H'(v'') < 0$.

By the strict MLRP we have that, for all $v \in [v_1, v_2]$ and all $v' > v$,

$$\begin{aligned} 0 &= \int_\eta^1 F\big(s^a(v,t)\big|v\big)\,dt - F\big(\alpha(v,\eta)\big|v\big) \\ &= \int_\eta^1 \int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)\,dx\,dt - \int_\eta^1 \int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)\,dx\,dt \\ &\leqslant \int_\eta^1 \int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}\,dx\,dt - \int_\eta^1 \int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}\,dx\,dt \\ &= \int_\eta^1 F\big(s^a(v,t)\big|v'\big)\,dt - (1-\eta)F\big(\alpha(v,\eta)\big|v'\big). \qquad \square \end{aligned}$$

**Proof of Proposition 2.** The proof is divided into two cases, depending on the value of $\eta$. For $\eta \in [0, \underline{\eta}]$, where $\underline{\eta} > 0$ is given by (32), I show that there is no mispricing; whereas when $\eta > \underline{\eta}$ there is mispricing. In the latter case, I provide an algorithm to find prices satisfying Proposition 1 and Corollary 1 and show that they exist and are unique. In addition, I show that, except for a very particular class of naive distributions, there exists $\overline{\eta} < 1$ such that for all $\eta \geqslant \overline{\eta}$ sophisticated bids are confined outside the range of equilibrium prices, implying that $\mathcal{V} = [0, 1]$.

Before turning into these cases, a prerequisite for existence is that any equilibrium prices satisfying Proposition 1 and Corollary 1 are in fact increasing. This is guaranteed if the mass of sophisticated bids is given by $F(\xi(b)|v)$ for some increasing function $\xi$ leads to market clearing prices that are also increasing. According to Proposition 1, market prices in mispricing intervals are given by (10):

$$1 - \gamma = \eta H\big(\rho(v)\big|v\big) + (1-\eta)F\big(s_k^*\big|v\big).$$

Given any $\eta \in (0, 1)$, the right-hand side of this expression is constant for $s_k^* \in \{0, 1\}$ and strictly increasing in $v$ for $s_k^* \in (0, 1)$. Hence, when $H(\cdot|\cdot)$ satisfies Assumption 4, the resulting price is increasing in $v$.

Now I turn into the two cases to be considered, $\eta \in [0, \underline{\eta}]$ and $\eta \in (\underline{\eta}, 1]$.

*Case 1*: ($\eta \in [0, \underline{\eta}]$). By Lemma 5, $\alpha(\cdot, \eta)$ is non-decreasing for all $\eta \leqslant \underline{\eta}$. This implies that there no mispricing in any monotone equilibrium for all $\eta < \underline{\eta}$, with the distribution of sophisticated bids satisfying $B^*(v|v) = F(\alpha(v, \eta)|v)$. In the absence of mispricing and since agents cannot affect the price, no sophisticated trader has an incentive to deviate and, hence, any profile of bidding strategies yielding $B^*$ constitutes a BNE. One such profile is given by $\beta(s, t) = \beta(s)$ for all $t$, with

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, \alpha(0, \eta)], \\ v \text{ s.t. } \alpha(v, \eta) = s & \text{if } s \in (\alpha(0, \eta), \alpha(1, \eta)), \\ 1 & \text{if } s \in [\alpha(1, \eta), 1]. \end{cases} \tag{33}$$

This takes care of existence of monotone equilibrium for $\eta \in [0, \underline{\eta}]$.

Regarding uniqueness of monotone equilibrium prices, assume there exists a monotone equilibrium with $\rho(v) \neq v$ a.e. in $(\underline{v}_1, \overline{v}_1)$ with $\underline{v}_1 < \overline{v}_1$ for some $\eta \leqslant \underline{\eta}$. If $\rho(\underline{v}_1) < \rho(\overline{v}_1)$, by

Lemma 3, the mass of sophisticated bids placed in $[\rho(\underline{v}_1), \rho(\bar{v}_1)]$ is zero. By Proposition 1 all sophisticated traders with signals below (above) some signal $s_1^*$ bid below $\rho(\underline{v}_1)$ (above $\rho(\bar{v}_1)$). However, given that $\alpha(\cdot, \eta)$ is increasing, we have that $s_1^* > \alpha(\underline{v}_1, \eta)$ and/or $s_1^* < \alpha(\bar{v}_1, \eta)$.[33] When $s_1^* > \alpha(\underline{v}_1, \eta)$ then $\rho(\underline{v}_1) < \underline{v}_1$ if $\underline{v}_1 > 0$ or $\rho(v) = 0$ in $[0, v')$ for some $v' > 0$ if $\underline{v}_1 = 0$, contradicting Corollary 1. On the other hand, if $s_1^* < \alpha(\bar{v}_1, \eta)$ then $\rho(\bar{v}_1) > \bar{v}_1$ if $\bar{v}_1 < 1$ or $\rho(v) = 1$ in $(v'', 1]$ for some $v'' < 1$ if $\bar{v}_1 = 1$, which again violates Corollary 1. Therefore, the only possibility left is that $\rho(v_1) = \rho(v_2)$, i.e. there exist an atom in the distribution of prices. But, according to Lemma 4, this can only happens when $\eta \geqslant \min\{\gamma, 1-\gamma\}$, i.e. when $\eta > \underline{\eta}$. Hence, when $\eta \in [0, \underline{\eta}]$ we have that $\rho(v) = v$ for all $v$.

*Case 2*: $(\eta \in (\underline{\eta}, 1])$. By part (ii) of Lemma 5, $\alpha(\cdot, \eta)$ is either non-monotonic or decreasing whenever it is well-defined. Hence, there exist a non-null set of asset values that are mispriced in equilibrium, given Lemma 6. The following algorithm identifies the values $\{\underline{v}_k\}_{k=1}^K$, $\{\bar{v}_k\}_{k=1}^K$ and signals $\{s_k^*\}_{k=1}^K$ that satisfy the conditions of Proposition 1 and Corollary 1. Then I show that these values and signals always exist and are unique. Finally, it is easy to check that the following symmetric bidding strategy implements equilibrium prices[34]:

$$\beta(s,t) = \begin{cases} 0 & \text{if } s < \min\{\alpha(0,\eta), s_1^*\}, \\ v \in [0, \underline{v}_1) \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in [\alpha(0,\eta), s_1^*), \\ v \in [\bar{v}_k, \underline{v}_{k+1}] \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in [s_k^*, s_{k+1}^*], \\ v \in (\bar{v}_K, 1] \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in (s_K^*, \alpha(1,\eta)], \\ 1 & \text{if } s > \max\{s_K^*, \alpha(1,\eta)\}. \end{cases} \tag{34}$$

The steps of the algorithm are:

1. Find asset values $\{v_i^m\}_{i=1}^I$ and $\{v_i^M\}_{i=1}^{I'}$ at which $\alpha(\cdot, \eta)$ reaches a local minimum and a local maximum, respectively. If $\alpha(\cdot, \eta)$ is not well-defined in an interval $(v', v'')$ with $\alpha(v', \eta)$ or $\alpha(v'', \eta) \in \{0, 1\}$, let $v'$ be the "unique" local maximum in $[v', v'']$ when $\alpha(v'', \eta) = 1$ and $v''$ be the "unique" local minimum when $\alpha(v', \eta) = 0$.[35] Let $v_0^m = 0$ and $v_{I'+1}^M = 1$.[36]
2. For each interval $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, with $j = 0$ if $v_1^m = 0$ and $j = 1$ if $v_1^M = 0$, find signal values $\{s_i\}_{i=1}^{I-1+j}$ such that, when $\rho(v)$ satisfies $1 - \gamma = \eta H(\rho(v)|v) + (1-\eta) F(s_i|v)$, are given by

$$s_i = \begin{cases} 0 & \text{if } \mathbb{E}((V - \rho(V)) 1_{\{V \in [v_{i-j}^m, v_{i-j+1}^m]\}} | 0) > 0, \\ 1 & \text{if } \mathbb{E}((V - \rho(V)) 1_{\{V \in [v_i^M, v_{i+1}^M]\}} | 1) < 0, \\ s & \text{if } \mathbb{E}((V - \rho(V)) 1_{\{V \in [\underline{v}_i(s), \bar{v}_i(s)]\}} | s) = 0, \end{cases} \tag{35}$$

where $\underline{v}_i(s)$, $\bar{v}_i(s)$ are respectively given by

$$\underline{v}_i(s) = \begin{cases} v_{i-j}^m & \text{if } \alpha(v_{i-j}^m, \eta) > s, \\ v \in [v_{i-j}^m, v_i^M] \text{ s.t. } \alpha(v,\eta) = s & \text{otherwise}, \end{cases} \tag{36}$$

[33] Note that $\alpha(\cdot, \eta)$ is strictly increasing for $\eta < \underline{\eta}$. If $\eta = \underline{\eta}$ and $\alpha(\cdot, \underline{\eta})$ is constant in $[\underline{v}_1, \bar{v}_1]$ then $s_1^* = \alpha(\underline{v}_1, \underline{\eta})$ would involve $\rho(v, \underline{\eta}) = v$ in $[\underline{v}_1, \bar{v}_1]$. Hence, one of these inequalities still needs to hold for $\rho(v, \underline{\eta}) \neq v$ a.e. in $[\underline{v}_1, \bar{v}_1]$.

[34] If $\mathcal{V}$ is the empty set, let $s_1^* = \alpha(1, \eta)$ and $\underline{v}_1 = 1$.

[35] Note that when $\alpha(v'', \eta) = 1$ either $\alpha(v', \eta) = 1$ or it is not well-defined. Similarly, when $\alpha(v', \eta) = 0$, when $\alpha(v'', \eta) = 0$ or it is not well-defined.

[36] By the continuity of $\alpha(\cdot, \eta)$, $v_i^m < v_i^M$ for all $i$ if $\alpha(0, \eta)$ is a local minimum, and $v_i^M < v_i^m$ for all $i$ if $\alpha(0, \eta)$ is a local maximum.

and

$$\overline{v}_i(s) = \begin{cases} v^M_{i+1} & \text{if } \alpha(v^M_{i+1}, \eta) < s, \\ v \in [v^m_{i-j+1}, v^M_{i+1}] \text{ s.t. } \alpha(v, \eta) = s & \text{otherwise.} \end{cases} \tag{37}$$

3. If $s_i > s_{i+1}$ merge intervals $[v^m_{i-j}, v^M_{i+1}]$ and $[v^m_{i+1-j}, v^M_{i+2}]$ and redefine $s_i = s'_i$ and $\overline{v}_i(s'_i) = \overline{v}_{i+1}(s'_i)$, with $s'_i$ given by

$$s'_i = \begin{cases} 0 & \text{if } \mathbb{E}((V - \rho(V))1_{\{V \in [v^m_{i-j}, v^m_{i-j+2}]\}}|0) > 0, \\ 1 & \text{if } \mathbb{E}((V - \rho(V))1_{\{V \in [v^M_i, v^M_{i+2}]\}}|1) < 0, \\ s & \text{if } \mathbb{E}((V - \rho(V))1_{\{V \in [\underline{v}_i(s), \overline{v}_{i+1}(s)]\}}|s) = 0. \end{cases} \tag{38}$$

   Repeat this step until $s_i \leqslant s_{i+1}$ for $i = 1, \ldots, K$, with $K$ being the new number of intervals.
4. Define $s^*_k = s_k$, $\underline{v}_k = \underline{v}_k(s_k)$ and $\overline{v}_k = \overline{v}_k(s_k)$, $k = 1, \ldots, K$.

Several things are worth noting. First, each interval $[v^m_{i-j}, v^M_{i+1}]$ contains $v^m_{i-j+1}$ and $v^M_i$. Thus, $\alpha(\cdot, \eta)$ is increasing in $(v^m_{i-j}, v^M_i) \cup (v^m_{i-j+1}, v^M_{i+1})$ and decreasing in $(v^M_i, v^m_{i-j+1})$.[37] This implies $s_i \in [\alpha(v^m_{i-j+1}, \eta), \max\{\alpha(v^M_i, \eta), \alpha(v^M_{i+1}, \eta)\}]$. Assume otherwise that $0 < s_i < \alpha(v^m_{i-j+1}, \eta) < 1$. Then $\rho(v) > v$ in $[\underline{v}_i(s_i), \overline{v}_i(s_i)]$ since $\alpha(v, \eta)$ is above $s_i$ in $[\underline{v}_i(s_i), v^M_{i+1}(s_i)]$. But then (35) would be violated since this leads to $\mathbb{E}((V - \rho(V))1_{\{V \in [\underline{v}_i(s_i), \overline{v}_i(s_i)]\}}|s_i) < 0$ when $s_i < 1$. Given these bounds on $s_i$, there is a unique value $v'_i(s_i) \in (v^M_i, v^m_{i-j+1})$ such that $\alpha(v'_i(s_i), \eta) = s_i$. Accordingly, $\rho(v) > v$ in $(\underline{v}_i(s_i), v'_i(s_i))$ and $\rho(v) < v$ in $(v'_i(s_i), \overline{v}_i(s_i))$.[38]

Second, $\underline{v}_i(\cdot)$ and $\overline{v}_i(\cdot)$ are increasing, while $v'_i(\cdot)$ is decreasing. By the continuity assumptions and Fact 1, each tuple $(s_i, \underline{v}_i(s_i), \overline{v}_i(s_i))$ exists and it is unique. To see why, note that as $s_i$ grows the interval where prices are above values $(\underline{v}_i(s_i), v'_i(s_i))$ shrinks while $(v'_i(s_i), \overline{v}_i(s_i))$ grows. Furthermore, as $s_i$ grows the probability mass (conditional on $s_i$) associated to $(v'_i(s_i), \overline{v}_i(s_i))$ grows relative to the mass associated to $(\underline{v}_i(s_i), v'_i(s_i))$, by the MLRP of $F(\cdot|s_i)$. Therefore, there is a unique signal $s_i$ (which in turn uniquely determines $\underline{v}_i(s_i)$ and $\overline{v}_i(s_i)$) satisfying (35).

Third, when two adjacent intervals with signals $s_i, s_{i+1}$ are merged (step 3 of the algorithm), the new pivotal signal $s'_i$ lies in $(s_{i+1}, s_i)$. Thus, any subinterval of $[\underline{v}_i(s'_i), \overline{v}_{i+1}(s'_i)]$ with $\rho(v) < v$ is preceded by a subinterval with $\rho(v) > v$, which means that we can apply the same existence and uniqueness argument to the tuple $(s'_i, \underline{v}_i(s'_i), \overline{v}_{i+1}(s'_i))$.

Finally, $\alpha(\cdot, \eta)$ is increasing in $[0, \underline{v}_1(s_1)]$, $[\overline{v}_i(s_i), \underline{v}_{i+1}(s_i)]$ and $[\underline{v}_K(s_K), 1]$. That is, it is increasing in $[0, 1] \setminus \bigcup_k [\underline{v}_k, \overline{v}_k]$, yielding $\rho(v) = v$ in such set (Lemma 6).

Given all of this, (35)–(38) imply that $\{(s^*_k, \underline{v}_k, \overline{v}_k)\}$ satisfy (7)–(9). Moreover, prices given by (10) are monotonic and satisfy Corollary 1.

Since this algorithm provides a unique solution, we need to show that a collection $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ not satisfying (35)–(38) violates (7)–(9) or Corollary 1.

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[37] Note that for $i = 0$, $v^m_{i-j} = v^M_i$ when $v^M_1 = 0$, and $v^m_{i-j+1} = v^M_{i+1}$ for $i = 1$ when $v^m_K = 1$.

[38] This is also true when $s_i \in \{0, 1\}$. Given (36)–(37), $s_i = 0$ implies that $\underline{v}_i = v^m_{i-j}$ and $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta} < 0$ in some interval $(v', v^m_i)$ (otherwise (35) would be violated), which leads to $\overline{v}_i = v^m_i$ (according to step 1 of the algorithm, $v^m_i$ is the upper bound of the interval of values where $\alpha(\cdot, \eta)$ is not well-defined). The latter implies that $\rho(v) < v$ in $(v', \overline{v}_i)$. Since $\alpha(\cdot, \eta)$ is either increasing in $(v^m_{i-j}, v^M_i)$ or above 0 when $v^m_{i-j} = 0$ (part (i) of Lemma 5), $\alpha(v, \eta) > 0$ (and thus $\rho(v) > v$) in $(\underline{v}_i, v')$. Similarly, $s_i = 1$ implies that $\underline{v}_i = v^M_i$ and $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta} > 1$ in some interval $(v^M_i, v')$, which means that $\overline{v}_i = v^M_{i+1}$. Hence, $\rho(v) > v$ in $(v^M_i, v')$ and $\rho(v) < v$ in $(v', v^M_{i+1})$.

Assume that there is a collection $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ satisfying Proposition 1. If $s'_h \in (0,1)$ then $\mathbb{E}((V-\rho(V))1_{\{V\in[\underline{v}'_h,\overline{v}'_h]\}}|s'_h)=0$ by (9). In addition, (7)–(8) and Corollary 1 require that $\alpha(\underline{v}'_h,\eta) \geqslant s'_h$ with equality when $\underline{v}'_h \in (0,1)$ and $\alpha(\overline{v}'_h,\eta) \leqslant s'_h$ with equality when $\underline{v}'_h \in (0,1)$. Corollary 1 further requires $\alpha(\cdot,\eta)$ to be increasing at $\underline{v}'_h$ and $\overline{v}'_h$ whenever it is equal to $s'_h$. All these conditions imply that $\underline{v}_h \in [v^m_{i-j}, v^M_i]$ and $\overline{v}_h \in [v^m_{l-j}, v^M_l]$ for some $i,l$ with $i<l$. But then, if $i=l+1$, $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s_i, \underline{v}_i, \overline{v}_i)$ given (35)–(38). On the other hand, if $i<l+1$ let $s_k$, $k=i,\dots,l$, be the signals given by (35). If $s_k < s_{k+1}$ for all $k$ then $s'_h \in (s_i, s_l)$ for $\mathbb{E}((V-\rho(V))1_{\{V\in[\underline{v}'_h,\overline{v}'_h\}}]|s'_h)=0$ to hold. But then $\mathbb{E}((V-\rho(V))1_{\{V\in[\underline{v}'_h,\overline{v}_i(s'_h)\}}]|s'_h)>0$ by Fact 1 and, in turn, $\mathbb{E}((V-\rho(V))1_{\{V\in[\overline{v}_i(s'_h),\overline{v}'_h\}}]|s'_h)<0$. Thus, a sophisticated trader receiving $s'_h$ would rather bid $\overline{v}_i(s'_h)$ than bid below $\underline{v}'_h$, contradicting that $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ correspond to equilibrium prices. Assume then that there exists some $i \leqslant h \leqslant l$ such that $s_h \geqslant s_{h+1}$. In such case, abusing notation, let $\{s_{i'}\}$ denote the new collection of signals, associated to intervals $[v^m_{i-j}, v^M_{i+1}]$ included in $[\underline{v}'_h, \overline{v}'_h]$, given by (38) after merging intervals $[v^m_{h-j}, v^M_{h+1}]$ and $[v^m_{h+1-j}, v^M_{h+2}]$. If $s_{i'} < s_{i'+1}$ for some $i'$ in the new collection of signals, we again have that $\mathbb{E}((V-\rho(V))1_{\{V\in[\overline{v}_{i'}(s'_h),\overline{v}'_h\}}]|s'_h)<0$, which leads to a profitable deviation by a trader receiving signal $s'_h$. By using this argument iteratively, we arrive at the conclusion that $(s_i, \underline{v}_i(s_i), \overline{v}_i(s_i))$ is the unique tuple satisfying (36)–(38), which are equivalent to (7)–(9), compatible with the equilibrium behavior of sophisticated traders. Hence, $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ cannot part of a characterization of equilibrium prices if $(s'_h, \underline{v}'_h, \overline{v}'_h) \neq (s_i, \underline{v}_i(s_i), \overline{v}_i(s_i))$.

If $s'_h = 0$ then $\underline{v}'_h = 0$ by part (i) of Lemma 5. We also have by (9) that $\mathbb{E}((V-\rho(V))1_{\{V\in[0,\overline{v}'_h\}}]|0) \geqslant 0$. In addition, Corollary 1 requires that $\rho(v)<v$ in the upper part of $[0,\overline{v}'_h)$, which means that $\overline{v}'_h \in [v^m_{i-j}, v^m_{i-j+1}]$ for some $i=1,\dots,I-1+j$. But this can only happen if $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}<0$ in some interval $(v', v^m_{i-j+1})$. Thus, $\overline{v}'_h = v^m_{i-j+1}$, otherwise $\overline{v}'_1$ would not satisfy (8). We need to consider two cases. If $i=1$ we have that the unique triplet satisfying these conditions is $(s_1, \underline{v}_1, \overline{v}_1)$ as defined by the above algorithm. If $i>1$ and $s_l < s_{l+1}$ for all $l<i$, then $\mathbb{E}((V-\rho(V))1_{\{V\in[\underline{v}_{l+1}(s_{l+1}),\overline{v}_{l+1}(s_{l+1})]\}}|s_{l+1})=0$ with $\underline{v}_{l+1}(s_{l+1}) < v^m_{i-j+1}$ (otherwise $\rho(v)<v$ a.e. in $[\underline{v}_{l+1}(s_{l+1}), \overline{v}_{l+1}(s_{l+1})]$) and a trader receiving a signal in $(s_l, s_{l+1})$ would rather deviate and bid $\underline{v}_i(s_{l+1})$. Therefore, $s_l > s_{l+1} = 0$ for some $l<i$. Using iterative merging we arrive at the conclusion that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_1, \underline{v}_1, \overline{v}_1)$ or that $(s'_h, \underline{v}'_h, \overline{v}'_h)$ violates Proposition 1.

Finally, when $s'_h = 1$ we have that $\overline{v}'_h = 1$ by part (i) of Lemma 5, and that $\mathbb{E}((V-\rho(V))1_{\{V\in[\underline{v}'_h,1]\}}]|1) \leqslant 0$ by (9). The latter implies that $\frac{1-\gamma-\eta\mathcal{H}(v)}{1-\eta}>1$ in an interval $(v^M_{i-j'}, v')$ for some $i=1,\dots,I'$ with $j'=0$ if $v^M_{I'}<1$ and $j'=1$ otherwise. Also, Corollary 1 requires that $\rho(v)>v$ in the lower part of $(\underline{v}'_h, 1]$. Hence, $\underline{v}'_h = v^M_i$ by (7). When $i=I'$, $(s_K, \underline{v}_K, \overline{v}_K)$ is the only triplet satisfying the above conditions. If $i<I'$ it has to be that $s_{l+1} < s_l = 1$ for some $l \geqslant i$, otherwise a trader with $s \in (s_l, s_{l+1})$ would deviate and bid $\overline{v}_{l+1}(s_{l+1}) > v^M_i$, given Fact 1 and that $\mathbb{E}((V-\rho(V))1_{\{V\in[\underline{v}_{l+1}(s_l),\overline{v}_l(s_l)]\}}|s_l)=0$. Therefore, using the above merging argument it has to be that $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_K, \underline{v}_K, \overline{v}_K)$, otherwise Proposition 1 would not hold. This completes the proof that a collection $\{(s^*_k, \underline{v}_k, \overline{v}_k)\}$ satisfying (7)–(9) exists and is unique.

To finish the proof of Proposition 2, we need to show that there exists $\overline{\eta}$ such that $\mathcal{V}=[0,1]$ for all $\eta \geqslant \overline{\eta}$. By Lemma 6, prices that equal values can only exist for values such that $\alpha(\cdot,\eta)$ is increasing. In addition, by Lemma 5, once $\alpha(\cdot,\eta)$ is decreasing at $v$ it is decreasing for all $\eta'>\eta$. Therefore, if there exists a share $\overline{\eta}$ such that $\alpha(\cdot,\eta)$ is either decreasing or not well-defined, it will also be so for all $\eta>\overline{\eta}$. In this context, the mass of bids at $[0,\rho(0)]$ (resp. $[\rho(1),1]$) is given by the mass of signals $s \leqslant s^*_1$ ($s > s^*_1$), where

$$s_1^* = \begin{cases} 0 & \text{if } \mathbb{E}(V - \rho(V)|s) > 0\ \forall s, \\ 1 & \text{if } \mathbb{E}(V - \rho(V)|s) < 0\ \forall s, \\ s \text{ s.t. } \mathbb{E}(V - \rho(V)|s) = 0 & \text{otherwise,} \end{cases} \tag{39}$$

with $\rho(v)$ satisfying $1 - \gamma = \eta H(\rho(v)|v) + (1 - \eta)F(s_i|v)$. The signal $s_1^*$ exists and it is unique as shown above.

Note that, by Lemma 5, if for any $v$ such that $\mathcal{H}(v) = 1 - \gamma$ we have that $\mathcal{H}'(v) > 0$, then there exists $\overline{\eta} < 1$ such that $\alpha(\cdot, \eta)$ is decreasing for all $\eta > \underline{\eta}$, leading to complete mispricing (Lemma 6).

If, however, $\mathcal{H}'(v) < 0$ for at least a value $v$ satisfying $\mathcal{H}(v) = 1 - \gamma$, a region without mispricing may exist around $v$ for all $\eta < 1$. To see why, note that for any such value there are two values $v'$ and $v''$ with $v' < v < v''$ such that $\mathcal{H}(v') = \mathcal{H}(v'') = 1 - \gamma$ and $\mathcal{H}(v'), \mathcal{H}(v'') > 0$. Accordingly, for $\eta$ close to one, $\alpha(\cdot, \eta)$ is decreasing in a neighborhood of $v'$ and $v''$ and increasing in a neighborhood of $v$, and by its continuity, its range in these neighborhoods is the whole unit interval. Thus there are at least two intervals $\{[v^m_{i-j}, v^M_{i+1}]\}_{i=1}^{I-1+j}$, $i = 1, 2$ as defined in the above algorithm. If the two signals satisfying (35) for each interval are such that $s_1 < s_2$, there exists a region with no mispricing in the interval $[\overline{v}(s_1), \underline{v}(s_2)]$ with $\underline{v}(s_2)$, $\overline{v}(s_1)$ given by (36) and (37), respectively.

Hence, if $\mathcal{H}'(v) > 0$ for any value $v$ such that $\mathcal{H}(v) = 1 - \gamma$, then $\overline{\eta} < 1$ whereas $\overline{\eta}$ might be equal to one otherwise. □

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