Online Appendix for *Naive Traders and Mispricing in Prediction Markets*: Examples.

Ricardo Serrano-Padial*
University of Wisconsin-Madison

January 31, 2012

This document provides two numerical examples that help illustrate how the extent of mispricing varies with the share of naive traders, and why, in expectation, similar mispricing patterns should happen in the presence of aggregate uncertainty about the fraction of naive traders in the market.

Example 2 *Consider a CVDA with the following characteristics. The distribution of signals conditional on* v *is given by* $Beta(1+v,1)$*, i.e.,* $F(s|v) = s^{1+v}$*. Each naive trader bids according to* $\beta^n(s) := \frac{3}{5}s^{1/5}$*, which is a rough approximation of bidding* $\mathbb{E}(V|s)$.[1]

Given $\beta^n(\cdot)$, the distribution of naive bids is

$$H(p|v) = \begin{cases} \left(\frac{5}{3}p\right)^{5(1+v)} & \text{if } v \leq \frac{3}{5} \\ 1 & \text{if } v > \frac{3}{5} \end{cases} \tag{1}$$

By *Proposition 2*, there exist cutoff points $\underline{\eta}$, $\overline{\eta}$ that determine whether there will be no, partial or complete mispricing as a function of η. Since $H'(v) \geq 0$ for all v, $\overline{\eta}$ is strictly less than one.

The first thing to note is that, given η, a necessary condition for partial mispricing with $\mathcal{V} = [\underline{v}_1, \overline{v}_1]$ is that there exist a signal s_1^* satisfying (7) at three distinct values, namely $\underline{v}_1$, $\overline{v}_1$ and $v_1' \in (\underline{v}_1, \overline{v}_1)$, the latter being the point at which $\rho(v)$ goes from being above to go below v. Therefore, the function $\alpha(v, \eta)$ given by

$$\alpha(v,\eta) = F^{-1}\left(\frac{1-\gamma-\eta H(v)}{1-\eta}|v\right) = \left[\frac{1-\gamma-\eta(1_{\{v>3/5\}} + 1_{\{v\leq 3/5\}}\left(\frac{5}{3}v\right)^{5(1+v)})}{1-\eta}\right]^{\frac{1}{1+v}}$$

*Email: rserrano@ssc.wisc.edu; Web: www.ssc.wisc.edu/~rserrano; Address: Department of Economics, Social Science Building, 1180 Observatory Drive, Madison, WI 53706-1393. Phone: (608) 239 4503. Fax: (608) 262 2033.

[1] This approximation makes computations more tractable without changing any substantive aspect of the analysis.

needs to be three-to-one in some subset of its range. If it is strictly increasing in $[0,1]$, then equilibrium prices will necessarily equal to values. On the other hand, if for some η there exists a signal s such that $\mathbb{E}(v - \rho(v)|s) = 0$ where $\rho(v)$ is given by $1 - \gamma = \eta H(\rho(v)|v) + (1-\eta)F(s|v)$ for all $v \in [0,1)$ and satisfies $\rho(0) > 0$ and $\rho(1) < 1$, then $[v_1, v_2] = [0,1]$ fulfils *Corollary 1*, and sophisticated bids will be confined to $[0,1] \setminus (\rho(0), \rho(1))$.[2]

In a symmetric market ($\gamma = 0.5$), I find that $\underline{\eta} \approx 0.016$ and $\overline{\eta} \approx 0.214$. This shows that the range of η compatible with perfect prices can be quite small. As an illustration, the top graph in *Figure A's* left panel shows equilibrium prices when 10% of traders are naïve. Even with such a low proportion of naive traders, the probability that prices reflect the true asset value is roughly one half in this example.

The graph of $\alpha(v, 0.1)$ (middle graph in the left panel of *Figure A* provides some intuition on the existence and uniqueness of prices. As mentioned above, $(s_1^*, \underline{v}_1, \overline{v}_1)$ are given by (7)-(8), that is $\rho(\underline{v}_1) = \underline{v}_1$, $\rho(\overline{v}_1) = \overline{v}_1$ and $\mathbb{E}((V - \rho(V)1_{\{v \in [\underline{v}_1, \overline{v}_1]\}}|s_1^*) = 0$. The latter implies that the expected gain a seller with signal s_1^* makes when she trades at $\rho(v) > v$ is exactly offset by trades at $\rho(v) < v$: these two regions are given by $[\underline{v}_1, v_1')$ and $(v_1', \overline{v}_1]$, respectively. Looking at the graph of $\alpha(v, 0.1)$ we can see that, as s_1^* increases, the distance between $\underline{v}_1$ and v_1' goes to zero implying that the set of trades with positive payoff shrinks to zero. Similarly, the distance between v_1' and $\overline{v}_1$ goes to zero when s_1^* decreases. Therefore, by the continuity of $\mathbb{E}(\cdot|\cdot)$ and $\alpha(\cdot, 0.1)$, we can find a unique triplet $(s_1^*, \underline{v}_1, \overline{v}_1)$ satisfying the conditions of *Proposition 1* and *Corollary 1*.

To complete the example, the bottom graph of *Figure A's* left panel shows symmetric equilibrium bidding strategies implementing $\rho(\cdot, 0.1)$.

Prices when the fraction of naive traders is unknown

Example 3 *Consider the CVDA of Example 2 with the random fraction of naive traders, denoted $\widetilde{\eta}$, distributed uniformly in $[0.05, 0.15]$ and independent of V.*

Let $\hat{\alpha}(p)$ be the quantile function such that $\mathbb{E}(V|\rho(V, \widetilde{\eta}) = p) = p$ for all $p \in [0,1]$, with $\rho(v, \eta)$ being the price when $V = v$, $\widetilde{\eta} = \eta$ and all sophisticated traders with signals below (above) $\hat{\alpha}(p)$ bid below (above) $\mathbb{E}(V|\rho(V, \widetilde{\eta} = p)$. In this example, $\hat{\alpha}(p)$ is non-monotonic, implying that in equilibrium there will be expected mispricing in some interval. This interval is given by the solution to the following system of

[2] If $\mathbb{E}(V - \rho(V)|0) \geq 0$ all the mass of risk-neutral bids would be placed above $\rho(1)$ whereas it would be placed below $\rho(0)$ when $\mathbb{E}(V - \rho(V)|1) \leq 0$.

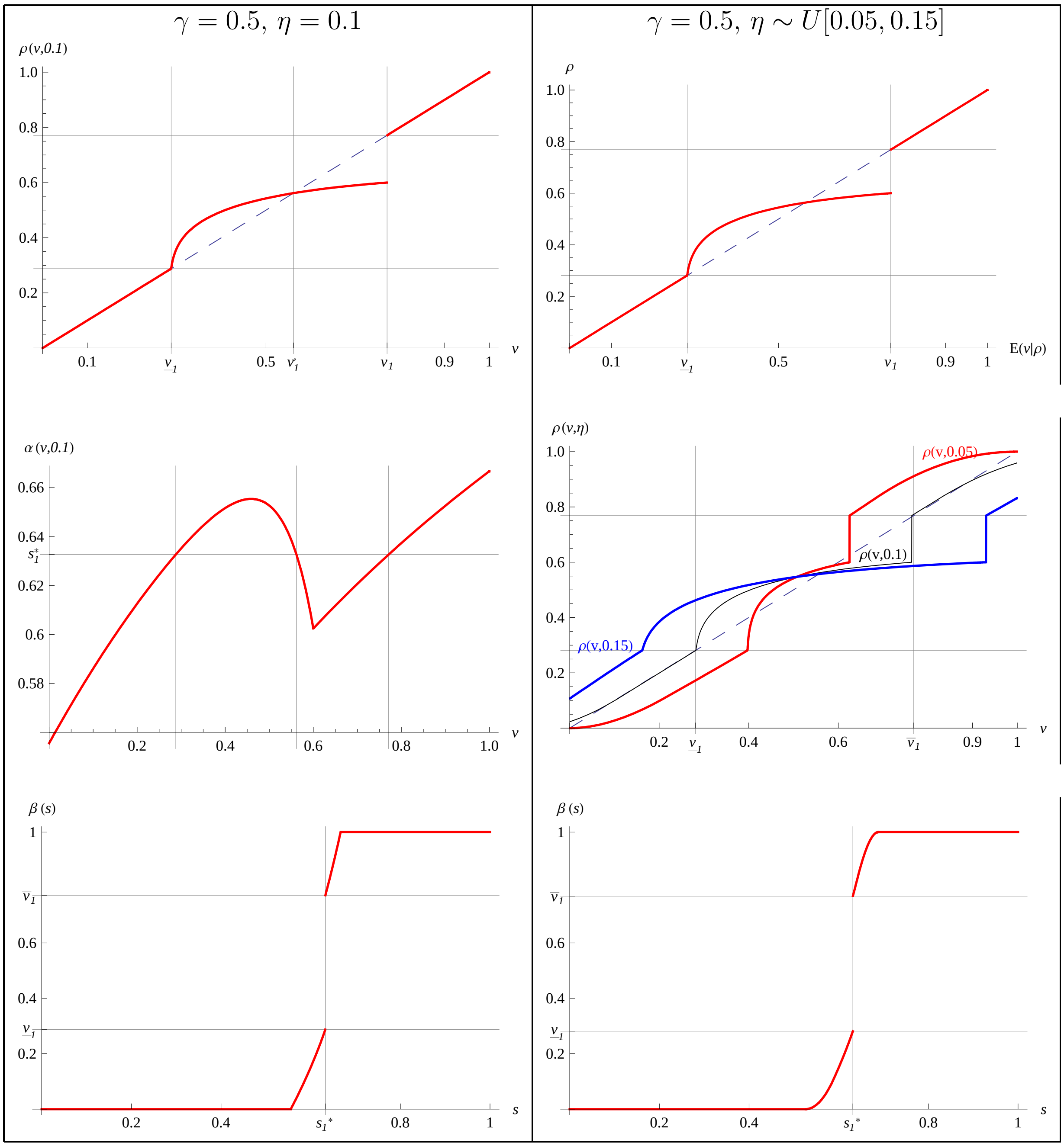

Figure A: Equilibrium prices and bidding strategies.

equations, which resembles the conditions in *Proposition 1*:

$$\begin{aligned}
&\mathbb{E}((V-\rho(V,\widetilde{\eta}))1_{\{\rho(V,\widetilde{\eta})\in[\underline{v}_1,\overline{v}_1]\}}|s_1^*)=0\\
&\mathbb{E}(V|\rho(V,\widetilde{\eta})=\underline{v}_1)=\underline{v}_1\\
&\mathbb{E}(V|\rho(V,\widetilde{\eta})=\underline{v}_2)=\underline{v}_2,
\end{aligned}$$

where $\rho(V,\widetilde{\eta})$ satisfies $1-\gamma=\widetilde{\eta}H(\rho(V,\widetilde{\eta})|V)+(1-\widetilde{\eta})F(s_1^*|V)$. The top graph in the right panel of *Figure A* shows the mapping from conditional expected values to prices, which is remarkably similar to prices when the fraction of naive traders is known and equal to 0.1. The middle graph shows prices for various realizations of $\widetilde{\eta}$. Equilibrium prices will lie in the area between $\rho(\cdot,0.05)$ and $\rho(\cdot,0.15)$. Interestingly, for low realizations of V, prices will be higher than values and viceversa (i.e. $\mathbb{E}(\rho(V,\widetilde{\eta})|v)>\ (<)\ v$ when v is close to zero (one)), even though there is no expected mispricing at low and high prices.[3]

Finally, the bottom graph depicts the discontinuous bidding strategy used by sophisticated traders in a symmetric equilibrium, which is given by

$$\beta(s)=\begin{cases}0 & \text{if } s<\frac{0.5}{0.95}\\ p\in[0,\underline{v}_1) \text{ s.t. } \hat{\alpha}(p)=s & \text{if } s\in\left[\frac{0.5}{0.95},s_1^*\right)\\ p\in(\overline{v}_1,1] \text{ s.t. } \hat{\alpha}(p)=s & \text{if } s\in\left[s_1^*,\sqrt{\frac{0.45}{0.95}}\right)\\ 1 & \text{if } s\geq\sqrt{\frac{0.45}{0.95}}.\end{cases} \tag{2}$$

[3]Actually, it is the requirement that $\mathbb{E}(V|\rho(V,\widetilde{\eta})=p)\leq p$ in equilibrium what causes it, given that if $\mathbb{E}(\rho(0,\widetilde{\eta})|0)=0$ then $\rho(0,\eta)=0$ for almost all η in the support and, by continuity of H and F, there exists a non-degenerate interval of values for which prices are zero with positive probability, implying that $\mathbb{E}(V|\rho(V,\widetilde{\eta})=0)>0$.